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Method for extracting arbitrarily large orbital equations of the Pincherle map

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ABSTRACT

We report an algorithm to extract equations of motion for orbits of arbitrarily high periods generated by iteration of the Pincherle map, the operational kernel used in the so-called *chaotic computers*. The performance of the algorithm is illustrated explicitly by extracting expeditiously, among others, an orbit buried inside a polynomial cluster of equations with degree exceeding one billion, out of reach by ordinary brute-force factorization. Large polynomial clusters are responsible for the organization of the phase-space and knowledge of this organization requires decomposing such clusters.

Introduction

Polynomials of arbitrarily large degrees arise naturally when algebraic equations of motion are iterated [1–3]. In such algebraic systems, knowledge of the structural organization of stability in phase space depends on the ability of factorizing polynomials of exceptionally large degrees which underlie all periodic orbits. The general aim of factorization consists of reducing something to its *basic building blocks*. In number theory, such building blocks are the prime numbers. In the algebra of polynomials, they are irreducible polynomial factors. The purpose of this paper is to introduce a method to factorize arbitrarily large polynomials generated by the equation of motion of a discrete-time dynamical system of algebraic origin, the Pincherle map [4,5], obtained by iterating the equation

$$x_{t+1} \equiv f(x_t) = 2 - x_t^2, \quad t = 0, 1, 2, \ldots \qquad (1)$$

Interest in this map was sparked by its key role as the kernel of a device proposed to perform general computations, namely the so-called *chaotic computer* [6–8]. There are also proposals of using Eq. (1) in multiple chaotic systems as a novel encryption technique, not yet free from problems [9]. Eq. (1) corresponds to selecting $a = 2$ in the quadratic map $x_{t+1} = a - x_t^2$, the parameter value well-known to yield fully developed chaos [10–12]. After a simple change of variable, Eq. (1) is equivalent to the celebrated and fruitful logistic map [10–12]. But the quadratic map involves less operations than the logistic map and, therefore, is more efficient for numerical studies. Eq. (1) plays also a significant role in multiparametric systems like, e.g., in the *canonical quartic map* [13,14] (Fig. 1). This map has often been used as a metaphor for a variety of phenomena in physics and involves the nonlinearity of lowest order normally present in series expansions of generic equations of motion. For details concerning Fig. 1 see the review in Ref. [15].

The method of factorization introduced here is based on the exploration of the preperiodic tails of orbits (Fig. 2), portions which are normally neglected in dynamics. Key to the method is our empirical observation that the preperiodic points of the orbits of Eq. (1) coincide with the zeros of a certain family of polynomials $T_\ell(x)$ which may be efficiently generated with a recurrence relation (Eq. (2) below). Thus, starting from the zeros of the polynomials $T_\ell(x)$, one may generate periodic orbits and their associated factors using simple forward iteration of the equation of motion, Eq. (1). As shown below explicitly, even rough numerical approximations of preperiodic points are sufficient to factor orbital equations exactly, allowing subsequent algebraic and arithmetic studies to be undertaken.

Our method does not involve the usual calculation of inverse functions. Inverse functions are multivalued and arguably the main factor that has prevented so far an effective exploration of preperiodic points. Computationally, methods involving inverse-functions

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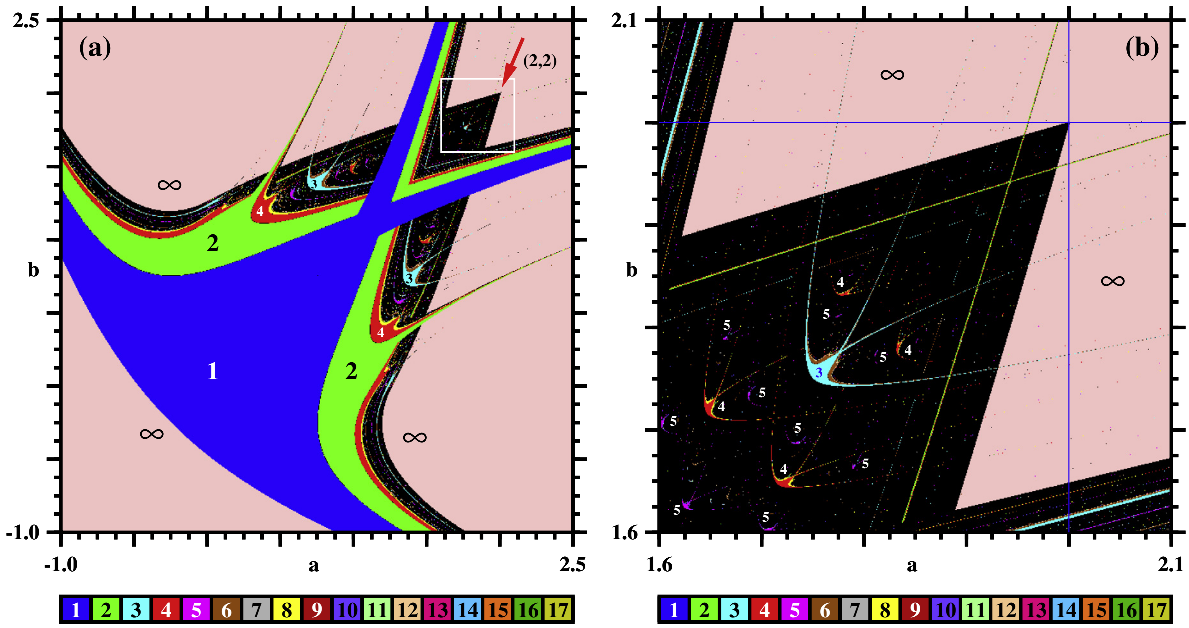

Fig. 1. The point $(a, b) = (2, 2)$ in the canonical quartic map $x_{t+1} = a - (b - x_t^2)^2$ [13,14] corresponds to the second iterate of Eq. (1). It is the double accumulation point located at the cusp formed by the intersection of the upper and lower boundaries between the black regions representing chaos and the pink regions denoting divergence of the iterative process. The myriad of colors and numbers indicate the period of the orbits in the region [15]. The exact identification of each stability island depends on polynomials whose degrees increase without bound with the period.

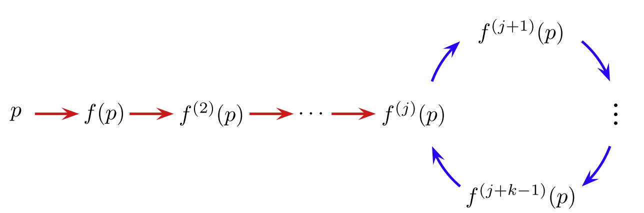

Fig. 2. Typical segments of the trajectory of a generic map $f(x)$ when starting from a preperiodic point p: After a *tail* of j iterates, the point lands on a k-periodic *cycle*. Distinct preperiodic points may land on different points of the cycle. The preperiodic and periodic segments contain a finite number of points. Analytically, arithmetic dynamics traces both segments exactly, without numerical approximations, because they are contained in the same number field. Numerically, the periodic cycle is approached only asymptotically. Each point of the cycle has its own set (tree) of preperiodic precursors.

become quickly impossible to implement due to an exponential or super-exponential proliferation and growth of the associated mathematical expressions and storage involved in their manipulation. In fact, to bypass this difficulty was our original motivation for trying the alternative reported here.

Before proceeding, we mention that it is possible to argue that exact orbits for Eq. (1) can be found using a conjugacy to the tent map, whose preperiodic orbits are proverbially related to a full shift on two symbols [12]. However, this conjugacy was not yet converted into practical applications, into investigations of the algebraic or arithmetic structure of the orbital equations, or into investigations of subtle nonlinear interdependencies that exist among them [16–18].

Nonlinear transformations: definitions and properties

We start by introducing the aforementioned polynomials $T_\ell(x)$ and some of their properties. Then, we illustrate the use of their zeros combined with forward iteration to factorize periodic cycles embedded in large polynomials generated by iterating the equation of motion, Eq. (1). As already mentioned, we have found empirically that the zeros of the polynomials $T_\ell(x)$ are preperiodic points of the orbits generated by Eq. (1).

The infinite set of transformations considered here may be generated recursively as follows:

$$T_\ell(x) = xT_{\ell-1}(x) - T_{\ell-2}(x), \quad \ell = 2, 3, 4, \ldots, \tag{2}$$

which depends on two initial functions: $T_1(x)$ and $T_0(x)$. Recurrences of this type are known as Fibonacci sequences. They are a simplified version of a more general one, depending on arbitrarily large sets of functions and parameters, which is of no concern in the present context. Here, we fix $T_1(x) = x$ and $T_0(x) = 2$. An alternative and direct way of obtaining $T_\ell(x)$ for a given ℓ is by using the relation [4]

$$T_\ell(x) = \left(\frac{x - \sqrt{x^2 - 4}}{2}\right)^\ell + \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^\ell, \quad \ell = 0, 1, 2, \ldots. \tag{3}$$

Polynomials related to $T_\ell(x)$ were associated to the names of Fibonacci, Lucas, Dickson, Chebyshev and possibly others, more recently in connection with the study of finite fields [19–21]. However, no applications to dynamical systems seem to exist yet.

Table 1 records explicitly the first few $T_\ell(x)$. With the help of Table 1 one realizes that some $T_\ell(x)$ may be obtained by commutative functional composition, e.g.

$$\begin{aligned} T_9(x) &= T_3^2(x) = T_3(T_3(x)) = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x, \\ &= (x^6 - 6x^4 + 9x^2 - 3)(x^2 - 3)x, \\ T_{15}(x) &= T_5(T_3(x)) = T_3(T_5(x)), \\ &= x^{15} - 15x^{13} + 90x^{11} - 275x^9 + 450x^7 - 378x^5 + 140x^3 - 15x, \\ &= (x^8 - 7x^6 + 14x^4 - 8x^2 + 1)(x^4 - 5x^2 + 5)(x^2 - 3)x. \end{aligned}$$

As seen from the above expressions, the polynomials $T_\ell(x)$ are not always irreducible but products of *cyclotomic-like* irreducible factors denoted here by $Q_k(x)$ (as defined in Tables 1 and 2). For instance, $T_9(x) = Q_9(x)Q_3(x)Q_1(x)$ and $T_{15}(x) = Q_{15}(x)Q_5(x)Q_3(x)Q_1(x)$. Every new $T_\ell(x)$ generated by Eq. (2) contains just a sigle new irreducible factor $Q_\ell(x)$, new in the sense of not appearing for any index smaller than ℓ. This property may be corroborated

Table 1
The first few orbital polynomials $T_\ell(x)$ defined by Eq. (2), together with their irreducible factors $Q_\ell(x)$, and their discriminants Δ_ℓ of the factors $Q_\ell(x)$. The polynomials are irreducible only for $\ell = 2^n, n = 0, 1, 2, \ldots$. Table 2 gives the decomposition of the first 100 polynomials $T_\ell(x)$.

ℓ	$T_\ell(x)$	$Q_\ell(x)$	Δ_ℓ
1	Q_1	x	1
2	Q_2	x^2-2	2^3
3	Q_1Q_3	x^2-3	$2^2 3^3$
4	Q_4	x^4-4x^2+2	2^{11}
5	Q_1Q_5	x^4-5x^2+5	$2^4 5^5$
6	Q_2Q_6	x^4-4x^2+1	$2^{11}3^6$
7	Q_1Q_7	$x^6-7x^4+14x^2-7$	$2^6 7^7$
8	Q_8	$x^8-8x^6+20x^4-16x^2+2$	2^{31}
9	$Q_1Q_3Q_9$	$x^6-6x^4+9x^2-3$	$2^8 3^{18}$
10	Q_2Q_{10}	$x^8-8x^6+19x^4-12x^2+1$	$2^{19}5^{10}$
11	Q_1Q_{11}	$x^{10}-11x^8+44x^6-77x^4+55x^2-11$	$2^{10}11^{11}$
12	Q_4Q_{12}	$x^8-8x^6+20x^4-16x^2+1$	$2^{35}3^{12}$
13	Q_1Q_{13}	$x^{12}-13x^{10}+65x^8-156x^6+182x^4-91x^2+13$	$2^{12}13^{13}$
14	Q_2Q_{14}	$x^{12}-12x^{10}+53x^8-104x^6+86x^4-24x^2+1$	$2^{27}7^{14}$
15	$Q_1Q_3Q_5Q_{15}$	$x^8-7x^6+14x^4-8x^2+1$	$2^{14}3^{15}5^{15}$
16	Q_{16}	$x^{16}-16x^{14}+104x^{12}-352x^{10}+660x^8-672x^6+336x^4-64x^2+2$	2^{79}
17	Q_1Q_{17}	$x^{16}-17x^{14}+119x^{12}-442x^{10}+935x^8-1122x^6+714x^4-204x^2+17$	$2^{16}17^{17}$
18	$Q_2Q_6Q_{18}$	$x^{12}-12x^{10}+54x^8-112x^6+105x^4-36x^2+1$	$2^{35}3^{36}$
19	Q_1Q_{19}	$x^{18}-19x^{16}+152x^{14}-665x^{12}+1729x^{10}-2717x^8+2508x^6-1254x^4+285x^2-19$	$2^{18}19^{19}$
20	Q_4Q_{20}	$x^{16}-16x^{14}+104x^{12}-352x^{10}+659x^8-664x^6+316x^4-48x^2+1$	$2^{59}5^{20}$
21	$Q_1Q_3Q_7Q_{21}$	$x^{12}-11x^{10}+44x^8-78x^6+60x^4-16x^2+1$	$2^{20}3^{21}7^{21}$
45	$Q_1Q_3Q_5Q_9Q_{15}Q_{45}$	$x^{24}-24x^{22}+252x^{20}-1519x^{18}+5796x^{16}-14553x^{14}+24206x^{12}$ $-26169x^{10}+17523x^8-6623x^6+1182x^4-72x^2+1,$	$2^{24}3^{36}5^{18}$

in Table 2 for the first 100 decompositions of $T_\ell(x)$. Many other properties are known [20] for the polynomials $T_\ell(x)$ which, however, are not needed here. Similarly to celebrated cyclotomic polynomials $(x^\ell - 1)/(x-1)$, the polynomials $T_\ell(x)$ are not irreducible (see Table 1). However, their basic building blocks $Q_\ell(x)$ are irreducible and interesting objects worth independent study.

Algorithm to obtain exact equations of motion

The key observation that we put into action now is the fact that all zeros of the $Q_\ell(x)$ (or, equivalently, of the $T_\ell(x)$) are preperiodic points (Fig. 2) of the equations of motion. Therefore, such zeros grant numerical (i.e. approximate) access to periodic orbits which, in their turn, lead to exact expressions for equations defining the orbits.

For a given value of ℓ, the extraction of exact periodic orbits consists of four steps:

1. Find numerical approximations of the zeros of $Q_\ell(x)$.
2. From such zeros, iterate the equation of motion, Eq. (1), to find numerical approximations for the points of a periodic orbit, say the points A_i.
3. Use the points A_i to determine numerically an approximate orbital equation, say $o_\ell(x) \equiv \prod_i^\ell (x - A_i)$.
4. Truncate every coefficient of $o_\ell(x)$ to the nearest integer to obtain the desired *exact* equation of motion, say $c_\ell(x)$, and, in principle, to exact expressions for all orbital points, namely the roots of $c_\ell(x)$.

This algorithm is illustrated below, where one sees that even modest numerical precision suffices to obtain exact equations of motion $c_\ell(x)$ and the corresponding zeros (orbital points).

Applications

We consider first the zeros for polynomials $Q_\ell(x)$ of lowest degrees, showing that they are preperiodic points. Iterating Eq. (1) from the initial point $x_0 = 0$, the zero of $Q_1(x)$, one sees that it is preperiodic and two iterates away from the fixed point $x = -2$. Similarly, the zeros $x = \pm\sqrt{2}$ of $Q_2(x) = 0$ are preperiodic, three iterates away from the same fixed point $x = -2$. The zeros $x = \pm\sqrt{3}$ of $Q_3(x)$ are preperiodic to the fixed point $x = 1$, two iterates from it. The four zeros $\pm\sqrt{2 \pm \sqrt{2}}$ of $Q_4(x)$ (approximated by ± 1.84775 and ± 0.765366) are preperiodic, four iterates away from $x = -2$.

Here are examples of less trivial periodic orbits. The four zeros of $Q_5(x)$ are two iterates away of a period-2 orbit oscillating indefinitely as follows:

$$\pm\frac{1}{2}\sqrt{10 - 2\sqrt{5}} \quad \text{enter the period-2 cycle} \quad A \to B \to A \to B \to \ldots,$$
$$\pm\frac{1}{2}\sqrt{10 + 2\sqrt{5}} \quad \text{enter the period-2 cycle} \quad B \to A \to B \to A \to \ldots,$$

where, iterating Eq. (1) from these zeros, one finds $A \simeq 1.618033$ and $B \simeq -0.618033$. From them it follows $o_2(x) = (x - 1.618033)(x + 0.618033) = x^2 - 1.000000x - 0.999997$. Truncating these coefficients one obtains the exact orbit $c_2(x) = x^2 - x - 1$ and exact representations for the orbital points: $A = (1 + \sqrt{5})/2$ and $B = (1 - \sqrt{5})/2$. That the zeros of $Q_k(x)$ are preperiodic points was verified numerically up to $k = 30$. No attempt was made to prove this in general.

Similarly as for the period-2 cycle above, other preperiodic points enter periodic cycles in a rigidly regular way. No point is missed and there are no repetitions, in the sense that preperiodic points with distinct magnitudes land on distinct points of the cycle. This implies a strict correlation, phase, between preperiodic and cycle points. For instance, the six (approximate) zeros of $Q_9(x)$ enter pairwise a period-3 cycle involving the points A, B, C:

$$\pm 0.68404 : A \to B \to C \to A \to \ldots,$$
$$\pm 1.28557 : B \to C \to A \to B \to \ldots,$$
$$\pm 1.96961 : C \to A \to B \to C \to \ldots,$$

where $A \simeq -0.347, B \simeq 1.879, C \simeq -1.532$ follow by iterating Eq. (1) from the six zeros above. From them, one gets

Table 2
Decomposition of the first 100 polynomials $T_i(x)$ in terms of its irreducible factors $Q_i(x)$. In this Table, "factor" is the factorization of i. Note the systemic regular interconnection between the prime factors of i and the subindexes of the irreducible factors of $T_i(x)$.

i	Factor	$T_i(x)$	Δ_i
1	1	Q_1	1
2	2	Q_2	2^3
3	3	Q_1Q_3	$2^2 3^3$
4	2^2	Q_4	2^{11}
5	5	Q_1Q_5	$2^4 5^3$
6	$2\cdot3$	Q_2Q_6	$2^8 3^2$
7	7	Q_1Q_7	$2^6 7^5$
8	2^3	Q_8	2^{31}
9	3^2	$Q_1Q_3Q_9$	$2^6 3^9$
10	$2\cdot5$	Q_2Q_{10}	$2^{16} 5^6$
11	11	Q_1Q_{11}	$2^{10} 11^9$
12	$2^2\cdot3$	Q_4Q_{12}	$2^{24} 3^4$
13	13	Q_1Q_{13}	$2^{12} 3^{11}$
14	$2\cdot7$	Q_2Q_{14}	$2^{24} 7^{10}$
15	$3\cdot5$	$Q_1Q_3Q_5Q_{15}$	$2^8 3^4 5^6$
16	2^4	Q_{16}	2^{79}
17	17	Q_1Q_{17}	$2^{16} 17^{15}$
18	$2\cdot3^2$	$Q_2Q_6Q_{18}$	$2^{24} 3^{18}$
19	19	Q_1Q_{19}	$2^{18} 19^{17}$
20	$2^2\cdot5$	Q_4Q_{20}	$2^{48} 5^{12}$
21	$3\cdot7$	$Q_1Q_3Q_7Q_{21}$	$2^{12} 3^6 7^{10}$
22	$2\cdot11$	Q_2Q_{22}	$2^{40} 11^{18}$
23	23	Q_1Q_{23}	$2^{22} 23^{21}$
24	$2^3\cdot3$	Q_8Q_{24}	$2^{64} 3^8$
25	5^2	$Q_1Q_5Q_{25}$	$2^{20} 5^{35}$
26	$2\cdot13$	Q_2Q_{26}	$2^{48} 13^{22}$
27	3^3	$Q_1Q_3Q_9Q_{27}$	$2^{18} 3^{45}$
28	$2^2\cdot7$	Q_4Q_{28}	$2^{72} 7^{20}$
29	29	Q_1Q_{29}	$2^{28} 29^{27}$
30	$2\cdot3\cdot5$	$Q_2Q_6Q_{10}Q_{30}$	$2^{32} 3^8 5^{12}$
31	31	Q_1Q_{31}	$2^{30} 31^{29}$
32	2^5	Q_{32}	2^{191}
33	$3\cdot11$	$Q_1Q_3Q_{11}Q_{33}$	$2^{20} 3^{10} 11^{18}$
34	$2\cdot17$	Q_2Q_{34}	$2^{64} 17^{30}$
35	$5\cdot7$	$Q_1Q_5Q_7Q_{35}$	$2^{24} 5^{18} 7^{20}$
36	$2^2\cdot3^2$	$Q_4Q_{12}Q_{36}$	$2^{72} 3^{36}$
37	37	Q_1Q_{37}	$2^{36} 37^{35}$
38	$2\cdot19$	Q_2Q_{38}	$2^{72} 19^{34}$
39	$3\cdot13$	$Q_1Q_3Q_{13}Q_{39}$	$2^{24} 3^{12} 13^{22}$
40	$2^3\cdot5$	Q_8Q_{40}	$2^{128} 5^{24}$
41	41	Q_1Q_{41}	$2^{40} 41^{39}$
42	$2\cdot3\cdot7$	$Q_2Q_6Q_{14}Q_{42}$	$2^{48} 3^{12} 7^{20}$
43	43	Q_1Q_{43}	$2^{42} 43^{41}$
44	$2^2\cdot11$	Q_4Q_{44}	$2^{120} 11^{36}$
45	$3^2\cdot5$	$Q_1Q_3Q_5Q_9Q_{15}Q_{45}$	$2^{24} 3^{36} 5^{18}$
46	$2\cdot23$	Q_2Q_{46}	$2^{88} 23^{42}$
47	47	Q_1Q_{47}	$2^{46} 47^{45}$
48	$2^4\cdot3$	$Q_{16}Q_{48}$	$2^{160} 3^{16}$
49	7^2	$Q_1Q_7Q_{49}$	$2^{42} 7^{77}$
50	$2\cdot5^2$	$Q_2Q_{10}Q_{50}$	$2^{80} 5^{70}$
51	$3\cdot17$	$Q_1Q_3Q_{17}Q_{51}$	$2^{32} 3^{16} 17^{30}$
52	$2^2\cdot13$	Q_4Q_{52}	$2^{144} 13^{44}$
53	53	Q_1Q_{53}	$2^{52} 53^{51}$
54	$2\cdot3^3$	$Q_2Q_6Q_{18}Q_{54}$	$2^{72} 3^{90}$
55	$5\cdot11$	$Q_1Q_5Q_{11}Q_{55}$	$2^{40} 5^{30} 11^{36}$
56	$2^3\cdot7$	Q_8Q_{56}	$2^{192} 7^{40}$
57	$3\cdot19$	$Q_1Q_3Q_{19}Q_{57}$	$2^{36} 3^{18} 19^{34}$
58	$2\cdot29$	Q_2Q_{58}	$2^{112} 29^{54}$
59	59	Q_1Q_{59}	$2^{58} 59^{57}$
60	$2^2\cdot3\cdot5$	$Q_4Q_{12}Q_{20}Q_{60}$	$2^{96} 3^{16} 5^{24}$
61	61	Q_1Q_{61}	$2^{60} 61^{59}$
62	$2\cdot31$	Q_2Q_{62}	$2^{120} 31^{58}$
63	$3^2\cdot7$	$Q_1Q_3Q_7Q_9Q_{21}Q_{63}$	$2^{36} 3^{54} 7^{30}$
64	2^6	Q_{64}	2^{447}
65	$5\cdot13$	$Q_1Q_5Q_{13}Q_{65}$	$2^{48} 5^{36} 13^{44}$
66	$2\cdot3\cdot11$	$Q_2Q_6Q_{22}Q_{66}$	$2^{80} 3^{20} 11^{36}$
67	67	Q_1Q_{67}	$2^{66} 67^{65}$
68	$2^2\cdot17$	Q_4Q_{68}	$2^{192} 17^{60}$
69	$3\cdot23$	$Q_1Q_3Q_{23}Q_{69}$	$2^{44} 3^{22} 23^{42}$
70	$2\cdot5\cdot7$	$Q_2Q_{10}Q_{14}Q_{70}$	$2^{96} 5^{36} 7^{40}$
71	71	Q_1Q_{71}	$2^{70} 71^{69}$
72	$2^3\cdot3^2$	$Q_8Q_{24}Q_{72}$	$2^{192} 3^{72}$
73	73	Q_1Q_{73}	$2^{72} 73^{71}$
74	$2\cdot37$	Q_2Q_{74}	$2^{144} 37^{70}$
75	$3\cdot5^2$	$Q_1Q_3Q_5Q_{15}Q_{25}Q_{75}$	$2^{40} 3^{20} 5^{70}$
76	$2^2\cdot19$	Q_4Q_{76}	$2^{216} 19^{68}$
77	$7\cdot11$	$Q_1Q_7Q_{11}Q_{77}$	$2^{60} 7^{50} 11^{54}$
78	$2\cdot3\cdot13$	$Q_2Q_6Q_{26}Q_{78}$	$2^{96} 3^{24} 13^{44}$
79	79	Q_1Q_{79}	$2^{78} 79^{77}$
80	$2^4\cdot5$	$Q_{16}Q_{80}$	$2^{320} 5^{48}$
81	3^4	$Q_1Q_3Q_9Q_{27}Q_{81}$	$2^{54} 3^{189}$
82	$2\cdot41$	Q_2Q_{82}	$2^{160} 41^{78}$
83	83	Q_1Q_{83}	$2^{82} 83^{81}$
84	$2^2\cdot3\cdot7$	$Q_4Q_{12}Q_{28}Q_{84}$	$2^{144} 3^{24} 7^{40}$
85	$5\cdot17$	$Q_1Q_5Q_{17}Q_{85}$	$2^{64} 5^{48} 17^{60}$
86	$2\cdot43$	Q_2Q_{86}	$2^{168} 43^{82}$
87	$3\cdot29$	$Q_1Q_3Q_{29}Q_{87}$	$2^{56} 3^{28} 29^{54}$
88	$2^3\cdot11$	Q_8Q_{88}	$2^{320} 11^{72}$
89	89	Q_1Q_{89}	$2^{88} 89^{87}$
90	$2\cdot3^2\cdot5$	$Q_2Q_6Q_{10}Q_{18}Q_{30}Q_{90}$	$2^{96} 3^{72} 5^{36}$
91	$7\cdot13$	$Q_1Q_7Q_{13}Q_{91}$	$2^{72} 7^{60} 13^{66}$
92	$2^2\cdot23$	Q_4Q_{92}	$2^{264} 23^{84}$
93	$3\cdot31$	$Q_1Q_3Q_{31}Q_{93}$	$2^{60} 3^{30} 31^{58}$
94	$2\cdot47$	Q_2Q_{94}	$2^{184} 47^{90}$
95	$5\cdot19$	$Q_1Q_5Q_{19}Q_{95}$	$2^{72} 5^{54} 19^{68}$
96	$2^5\cdot3$	$Q_{32}Q_{96}$	$2^{384} 3^{32}$
97	97	Q_1Q_{97}	$2^{96} 97^{95}$
98	$2\cdot7^2$	$Q_2Q_{14}Q_{98}$	$2^{168} 7^{154}$
99	$3^2\cdot11$	$Q_1Q_3Q_9Q_{11}Q_{33}Q_{99}$	$2^{60} 3^{90} 11^{54}$
100	$2^2\cdot5^2$	$Q_4Q_{20}Q_{100}$	$2^{240} 5^{140}$

$o_3(x) = (x-A)(x-B)(x-C) = x^3 - 2.9990x - 0.9988$. Thus, the exact equation for the period-3 cycle is $c_3(x) = x^3 - 3x - 1 = 0$, something trivial to corroborate. Analogously, iterating Eq. (1) one finds the eight zeros of $Q_{15}(x)$ to be preperiodic, reaching a period-4 cycle in the following order (phases):

$$\pm 0.4158233 : A \to B \to C \to D \to A \to \dots,$$
$$\pm 0.8134732 : B \to C \to D \to A \to B \to \dots,$$
$$\pm 1.4862896 : C \to D \to A \to B \to C \to \dots,$$
$$\pm 1.9890437 : D \to A \to B \to C \to D \to \dots.$$

As before, from the approximate values of A, B, C, D we find

$$\begin{aligned} o_4(x) &= (x-A)(x-B)(x-C)(x-D) \\ &= x^4 + 1.00001x^3 - 3.99997x^2 - 4.00001x + 0.99996, \end{aligned} \tag{4}$$

and the exact orbital equation $c_4(x) = x^4 + x^3 - 4x^2 - 4x + 1$ and exact representations of the period-4 points:

$$A = -\frac{1}{4}(1-\sqrt{5}) - \frac{1}{4}\sqrt{30+6\sqrt{5}} \simeq -1.33826, \tag{5}$$

$$B = -\frac{1}{4}(1+\sqrt{5}) + \frac{1}{4}\sqrt{30-6\sqrt{5}} \simeq 0.20905, \tag{6}$$

$$C = -\frac{1}{4}(1-\sqrt{5}) + \frac{1}{4}\sqrt{30+6\sqrt{5}} \simeq 1.95629, \tag{7}$$

$$D = -\frac{1}{4}(1+\sqrt{5}) - \frac{1}{4}\sqrt{30-6\sqrt{5}} \simeq -1.82709. \tag{8}$$

These results show that, instead of jumping randomly in phase space, the orbit cycles systematically between specific sets of (field) conjugate points, in a quite regular and prescribed sequence.

The next example reveals a remarkable interconnection of the period-4 orbit, Eqs. (5)–(8), and the 24 zeros of $Q_{45}(x)$ (Table 1). After two iterates such zeros enter a period-12 cycle *ABCDEFGHIJKLA*..., in consecutive points of the cycle in the following order: ±0.139, ±1.696, ±0.551, ±1.059, ±1.797, ±1.576, ±1.940, ±0.938, ±1.658, ±1.854, ±1.389, ±1.998. From these zeros, iterating Eq. (1) one finds the following approximations for the 12 orbital points: $A = -1.9225$, $B = -1.6960$, $C = -0.8767$, $D = 1.2313$, $E = 0.4838$, $F = 1.7658$, $G = -1.1183$, $H = 0.7492$, $I = 1.4386$, $J = -0.0697$, $K = 1.9951$, $L = -1.9805$. As before for Eq. (4), these rough approximations are enough to obtain the exact period-12 cycle:

$$c_{12}(x) = x^{12} - 12x^{10} + x^9 + 54x^8 - 9x^7 - 112x^6 + 27x^5 + 105x^4 - 31x^3 - 36x^2 + 12x + 1.$$

This equation factors into conjugate cubics, $c_{12}(x) = k_1(x) \cdot \overline{k_1(x)} \cdot k_2(x) \cdot \overline{k_2(x)}$ where

$$k_1(x) = x^3 - 3x + \tfrac{1}{4}(1-\sqrt{5}) + \tfrac{1}{4}\sqrt{30+6\sqrt{5}},$$
$$\overline{k_1(x)} = x^3 - 3x + \tfrac{1}{4}(1+\sqrt{5}) - \tfrac{1}{4}\sqrt{30-6\sqrt{5}},$$
$$k_2(x) = x^3 - 3x + \tfrac{1}{4}(1-\sqrt{5}) - \tfrac{1}{4}\sqrt{30+6\sqrt{5}},$$
$$\overline{k_2(x)} = x^3 - 3x + \tfrac{1}{4}(1+\sqrt{5}) + \tfrac{1}{4}\sqrt{30-6\sqrt{5}}.$$

From them follow exact representations for the period-12 cycle, namely for the triplets of orbital points (A,E,I), (B,F,J), (C,G,K), (D,H,L) obtained above as approximate numbers. Again, rough approximations of the zeros brought the equation of motion exactly and expeditiously. Arithmetically, the four factors above "descend" from the orbital points of the period-4 cycle in Eqs. (5)–(8). While such orbital inheritance is trivial to recognize from the exact representations, such subtle orbital interconnection is impossible to infer from the numerical values generated by iteration of the equation of motion.

Although conceptually simple, the brute-force generation of the polynomials by iteration is a computationally difficult problem that quickly becomes prohibitively hard, already for low periods, even when employing massive computational resources. Polynomials generated by brute-force for large periods k are gigantic, of very large degrees [22], and mix together wanted and unwanted zeros, the divisors of k. For example, for period-12 brute-force produces a polynomial mixing together the zeros of all 335 period-12 orbits possible in addition to the zeros of the 17 non-genuine orbits, i.e. orbits with lower periods which are divisors of 12 [22]. The degree of such polynomial is not smaller than $335 \times 12 = 4020$, and has very large coefficients. Once generated, this polynomial still needs to be factored in order to deliver $c_{12}(x)$ as one among its irreducible factors. In contrast, the polynomials $Q_\ell(x)$ allow one to obtain exact expressions for orbits of very long periods using modest resources.

As a first example of a long periodic orbit, we consider the period-20 orbit generated with the zeros of $Q_{55}(x)$, the polynomial of degree 40 associated with $T_{55}(x) = T_{11}(T_5(x))$ (Table 1). In this case, brute force would require extracting it from a big polynomial containing simultaneously the zeros of all the 52377 period-20 orbits together with zeros of the 111 orbits of their divisors [22], a polynomial of degree well above 10^6. In contrast, using the 40 zeros of $Q_{55}(x)$ and their iterates, the above procedure leads to a period-20 cycle that can be decomposed over $\mathbb{Q}(\sqrt{5})$:

$$\begin{aligned} c_{20}(x) = {} & x^{20} + x^{19} - 20x^{18} - 19x^{17} + 170x^{16} + 151x^{15} \\ & - 801x^{14} - 650x^{13} + 2289x^{12} + 1639x^{11} - 4080x^{10} \\ & - 2442x^9 + 4489x^8 + 2058x^7 - 2891x^6 - 877x^5 \\ & + 951x^4 + 151x^3 - 108x^2 - 12x + 1. \end{aligned} \tag{9}$$

The next example involves a curious situation where two distinct sets of preperiodic points land on the same orbit. Such preperiodic points are zeros of $Q_{77}(x)$ and $Q_{99}(x)$ extracted from $T_{11}(T_7(x))$ and $T_{11}(T_9(x))$ and given explicitly in the Appendix, Eqs. (10) and (11). They both lead to a period-30 cycle, Eq. (12) in the Appendix.

Is it possible to extract $c_{30}(x)$ by brute force, from the polynomial obtained by forward iteration of the equation of motion? To check this, we note that, altogether, there are 35790267 genuine period-30 orbits [22]. Thus, even neglecting all non-genuine orbits (those with periods given by divisors of 30), this would mean having to extract the orbit from manipulations of a polynomial *compositum* of equations having degree not lower than $35790267 \times 30 = 1073708010 \simeq 10^9$. This degree shows brute force to be a computationally unrealistic approach. Think of the size of its coefficients and the space needed to store them. Thus, it seems fair to say that preperiodic orbits allow one to reach otherwise unreachable orbits.

As a final example, we mention a historical fact concerning $T_{45}(x)$, apparently one of the earliest recorded use of a polynomial $T(x)$. In 1593, the Belgian mathematician Adriaan van Roomen challenged "all mathematicians of the world" [23–28] to solve some equations involving the polynomial $R(x)$ given explicitly in Eq. (13) of the Appendix. The challenge was presented to King Henry IV by a Dutch ambassador, who purportedly added that there was no mathematician of any importance in France able to solve the problem [23–28]. The King then summoned Viète (latin form: *Vieta*) to his presence who, confronted with the problem, *ut legi, ut solvi* [24], i.e. as soon as reading it provided one solution (which one?) on the spot presenting 22 additional ones the next day. From the factor listed in Table 1 it is not difficult to recognize that van Roomen's $R(x)$ is nothing else than $T_{45}(x)$, the polynomial leading to the period-12 cycle in Eq. (4), among other cycles of lower periods. Thus, $R(x)$ is just one among an infinite number of polynomials that could have been used to pose a host of intricate problems. Here, however, the twist is to recover $R(x)$ *dynamically* among the orbits of the logistic equation, an equation known to underly numerous practical applications [10–12,29,30].

The interplay between approximate and exact numbers, used here to reach orbits precisely, is connected to an insightful remark of Born. In his work "Is classical mechanics in fact deterministic?" Born writes [31]: "*A statement like $x = \pi$ cm would have a physical meaning if one could distinguish between it and $x = \pi_n$ cm for every n, where π_n is the approximation of π by the first n decimals. This however, is impossible; and even if we suppose that the accuracy of measurements will be increased in the future, n can always be chosen so large that no experimental distinction is possible. Of course I do not intend to banish from physics the idea of a real number. It is indispensable for the application of analysis. What I mean is that a physical situation must be described by means of real numbers in such a way that the natural uncertainty in all observations is taken into account.*" In the present context, Abelian equations, which underlie necessarily all periodic dynamics and the phase-space of a large class of dynamical systems, provide an efficient algorithm allowing rough "experimental" (approximate) orbital points to reveal expeditiously exact factors and arbitrarily precise dynamics, limited exclusively by the ability of discerning numbers that represent physical results.

Conclusions and outlook

In this paper, we reported an efficient computational method to obtain two significant pieces of information concerning the dynamical properties of Eq. (1), a paradigmatic map underlying the operating kernel of the so-called *chaotic computer* [6–8]: (i) to extract exact expressions for orbital equations $c_\ell(x)$ using preperiodic points, zeros of the irreducible polynomials $Q_\ell(x)$, and forward iteration; (ii) to grant access to periodic orbits of very high periods, orbits not accessible by a brute-force approach. The zeros of $Q_\ell(x)$ provide preperiodic points directly, without the need of chains of multivalued inverse functions of the equations of motion, a procedure that quickly becomes computationally impracticable. For reference, all computations reported were done within a few minutes with MAPLE running on a 8-years old LINUX operated Notepad X200s notebook.

An interesting open problem is to correlate the decomposition of orbital equations and the relative phase of the individual preperiodic segments. We already obtained orbits for periods much higher than the ones reported here. Such orbits can be classified into specific families, according to the nature of their algebraic structure, i.e. according to intricate relationships among the solutions of the equations of motion. The arithmetic properties of the polynomials $Q_\ell(x)$ depend on the prime numbers involved in the decomposition of ℓ. An open challenge is to find means of performing such classification in a purely algebraic way, i.e. without the need of computing orbits explicitly. Another enticing but certainly demanding task is to find a procedure to sort out more complex orbitals *clusters*, having integer coefficients but entangling more than one orbit simultaneously. This is tantamount to factor orbits over more complicated number fields, not over integers as done in the present paper. Furthermore, it would be interesting to count of number of polynomials $Q_\ell(x)$ sharing a fixed degree and investigate their interconnections with the orbital structure. Knowledge of the inner arithmetic scaffold underlying equations of motion might provide a bridge to understand the still very fragmentary facts connected with the *origins* of periodicity in dynamical systems [32].

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Appendix

This appendix presents explicit characterizations of results discussed in Section "Nonlinear transformations: definitions and properties". The irreducible factors $Q_\ell(x)$ shown in Table 1 may be split in suitable algebraic number fields using standard algebraic methods [33]. Explicitly, we present below results allowing one to use familiar formulas to derive exact representations for all periodic orbital points for ℓ up to 12, inclusive. For clarity, the listing includes both field conjugate factors. Multiple factorizations are also indicated.

$$
\begin{aligned}
Q_4(x) &= (x^2-2+\sqrt{2})(x^2-2-\sqrt{2}),\\
Q_5(x) &= (x^2+\tfrac{1}{2}(-5+\sqrt{5}))(x^2+\tfrac{1}{2}(-5-\sqrt{5})),\\
Q_6(x) &= (x^2+\sqrt{2}x-1)(x^2-\sqrt{2}x-1)\\
&= (x^2-2+\sqrt{3})(x^2-2-\sqrt{3})\\
&= (x^2+\sqrt{6}x+1)(x^2-\sqrt{6}x+1),\\
Q_7(x) &= (x^3+\sqrt{7}x^2-\sqrt{7})(x^3-\sqrt{7}x^2+\sqrt{7}),\\
Q_8(x) &= (x^4-4x^2+2+\sqrt{2})(x^4-4x^2+2-\sqrt{2}),\\
Q_9(x) &= (x^3-3x+\sqrt{3})(x^3-3x-\sqrt{3}),\\
Q_{10}(x) &= (x^4+\sqrt{2}x^3-3x^2-3\sqrt{2}x-1)(x^4-\sqrt{2}x^3-3x^2+3\sqrt{2}x-1),\\
&\quad (x^4-4x^2+\tfrac{1}{2}(3+\sqrt{5}))(x^4-4x^2+\tfrac{1}{2}(3-\sqrt{5})),\\
&\quad (x^4+\sqrt{10}x^3+x^2-\sqrt{10}x-1)(x^4-\sqrt{10}x^3+x^2+\sqrt{10}x-1),\\
Q_{11}(x) &= (x^5+\sqrt{11}x^4-3\sqrt{11}x^2-11x-\sqrt{11})\\
&\quad \times(x^5-\sqrt{11}x^4+3\sqrt{11}x^2-11x+\sqrt{11}),\\
Q_{12}(x) &= (x^4-(4+\sqrt{2})x^2+3+2\sqrt{2})(x^4-(4-\sqrt{2})x^2+3-2\sqrt{2}),\\
&= (x^4-4x^2+2+\sqrt{3})(x^4-4x^2+2-\sqrt{3}),\\
&= (x^4-(4+\sqrt{6})x^2+5-\sqrt{6})(x^4-(4-\sqrt{6})x^2+5+\sqrt{6}).
\end{aligned}
$$

Note that multiple decompositions of $Q_\ell(x)$ are intrinsically connected with the factorization of ℓ itself. With exception of the conjugate pair of quintics in $Q_{11}(x)$, the familiar expressions for the roots of cubic and quartic equations may be used to represent all roots exactly. Although not used explicitly here, we mention that we were able to obtain exact expressions in closed form for the zeros of the quintics, as functions of the following pair of irrationalities:

$$
\begin{aligned}
\alpha &= \sqrt{55+2\sqrt{5}},\\
\beta &= \sqrt[5]{(-109+25\sqrt{5})\alpha\sqrt{11}+(6325+4455\sqrt{5})\sqrt{-1}}.
\end{aligned}
$$

In all cases considered, the zeros of $Q_k(x)$ were invariably found to be preperiodic points.

The explicit expressions of $Q_{77}(x)$ and $Q_{99}(x)$ mentioned in the text are the following:

$$
\begin{aligned}
Q_{77}(x) = x^{60} &- 59x^{58} + 1652x^{56} - 29205x^{54} + 365859x^{52}\\
&- 3455335x^{50} + 25556440x^{48} - 151794021x^{46}\\
&+ 736647495x^{44} - 2956412711x^{42} + 9894941476x^{40}\\
&- 27773378270x^{38} + 65592924343x^{36} - 130530017729x^{34}\\
&+ 218801812945x^{32} - 308337579027x^{30}\\
&+ 363972489209x^{28} - 357976286928x^{26}\\
&+ 291226209171x^{24} - 194130948828x^{22}\\
&+ 104766369144x^{20} - 45083463663x^{18}\\
&+ 15176439088x^{16} - 3900841911x^{14} + 742221909x^{12}\\
&- 100422335x^{10} + 9156888x^{8} - 521938x^{6}\\
&+ 16580x^{4} - 240x^{2} + 1,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
Q_{99}(x) = x^{60} &- 60x^{58} + 1710x^{56} - 30799x^{54} + 393471x^{52}\\
&- 3793635x^{50} + 28674800x^{48} - 174252876x^{46}\\
&+ 866211786x^{44} - 3565385550x^{42} + 12254475528x^{40}\\
&- 35370320544x^{38} + 86022282164x^{36}\\
&- 176542368876x^{34} + 305660799936x^{32}\\
&- 445607921728x^{30} + 545048561739x^{28}\\
&- 556374246012x^{26} + 470519579611x^{24}\\
&- 326521492629x^{22} + 183667273857x^{20}\\
&- 82439284584x^{18} + 28942795476x^{16} - 7746369957x^{14}\\
&+ 1528528872x^{12} - 212659083x^{10} + 19610019x^{8}\\
&- 1094107x^{6} + 31830x^{4} - 360x^{2} + 1.
\end{aligned}
\tag{11}
$$

The discriminant of $Q_{77}(x)$ is $\Delta_{77} = 2^{60}7^{50}11^{54}$ and it decomposes in three distinct ways, over either $\mathbb{Q}(\sqrt{7}), \mathbb{Q}(\sqrt{11})$, or $\mathbb{Q}(\sqrt{77})$. The discriminant of $Q_{99}(x)$ is $\Delta_{99} = 2^{60}3^{90}11^{54}$ and, similarly, it decomposes over either $\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{11})$, or $\mathbb{Q}(\sqrt{33})$. Such multiple decompositions seem to show that these cyclotomic-like factors "remember" their arithmetical origin. The zeros of Eqs. (10) and (11) lead to the same period-30 cycle:

$$\begin{aligned} c_{30}(x) = {} & x^{30} + x^{29} - 30x^{28} - 29x^{27} + 405x^{26} + 377x^{25} - 3250x^{24} \\ & - 2901x^{23} + 17249x^{22} + 14697x^{21} - 63734x^{20} \\ & - 51590x^{19} + 168035x^{18} + 128611x^{17} - 318629x^{16} \\ & - 229651x^{15} + 432159x^{14} + 292608x^{13} - 411241x^{12} \\ & - 261924x^{11} + 264472x^{10} + 159873x^{9} - 107406x^{8} \\ & - 63143x^{7} + 24051x^{6} + 14609x^{5} - 2010x^{4} - 1562x^{3} \\ & - 72x^{2} + 24x + 1. \end{aligned} \tag{12}$$

Its discriminant is $7^{25}11^{27}$ and it is decomposable over $\mathbb{Q}(\sqrt{77})$. Here, the dynamics apparently filtered out the dependence over $\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{7})$, and $\mathbb{Q}(\sqrt{11})$, present in the preperiodic tail.

Here is the famous [23–28] van Roomen's polynomial $R(x)$ discussed in the text.

$$\begin{aligned} R(x) \equiv {} & x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} \\ & - 740259x^{35} + 3764565x^{33} - 14945040x^{31} \\ & + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} \\ & - 378658800x^{23} + 483841800x^{21} - 488494125x^{19} \\ & + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} \\ & - 34512075x^{11} + 7811375x^{9} - 1138500x^{7} \\ & + 95634x^{5} - 3795x^{3} + 45x. \end{aligned} \tag{13}$$

As indicated in Table 1, this polynomial is identical to $T_{45}(x)$, whose dynamics is discussed in details in Section "Applications". Table 2 provides a tabulation of the decomposition of the first 100 polynomials $T_\ell(x)$.

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