Coefficient of restitution of aspherical particles

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We consider the motion of an aspherical inelastic particle of dumbbell type bouncing repeatedly on a horizontal flat surface. The coefficient of restitution of such a particle depends not only on material properties and impact velocity but also on the angular orientation at the instant of the collision whose variance is considerable, even for small eccentricity. Assuming random angular orientation of the particle at the instant of contact we characterize the measured coefficient of restitution as a fluctuating quantity and obtain a wide probability density function including a finite probability for negative values of the coefficient of restitution. This may be understood from the partial exchange of translational and rotational kinetic energy.

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I. INTRODUCTION

Many-particle systems of dissipatively colliding hard spheres may be theoretically described by kinetic theory based on the approximative solution of the Boltzmann equation (e.g., [1]). Corresponding numerical simulations compute either the detailed particle trajectories via event-driven molecular dynamics (e.g., [2–4] or compute the spatiotemporal evolution of the velocity distribution function via direct simulation Monte Carlo (e.g., [3,5,6]). All three approaches rely on the coefficient of restitution relating the normal component of the relative velocity before and after an impact.

Assuming perfect spheres, the coefficient of restitution can be obtained by solving Newton's equation of motion for the central collision of a pair of particles or, equivalently, for the collision of a particle with a solid wall. This way, we obtain the coefficient of restitution as a (deterministic) function of material characteristics, particle sizes, and impact velocity [7–9].

If we wish to extend kinetic theory and event-driven molecular dynamics to nonspherical particles, we need a generalization of the ordinary coefficient of restitution. For nonspherical particles, however, the coefficient of restitution depends not only on material characteristics, particle sizes, and impact velocity but, moreover, on the angular orientation of the particles at the instant of the collision. In many cases, in particular for dilute gases, successive collisions of particles are only weakly correlated [1], which suggests one considers the coefficient of restitution as a random variable with certain statistical characteristics describing the fluctuations. In a first attempt in this direction, rough particles have been modeled as spheres whose surface is covered by a large number of randomly distributed asperities [10]. Here, the coefficient of restitution becomes a fluctuating quantity as its value depends on the details of the surface at the contact point, that is, on the random geometry of the collision. It was shown that this model leads to a characteristic probability distribution [11] which agrees well with large-scale experiments [10].

This result obtained for imperfect (rough) spheres suggests the investigation of collisions of particles which may be also considered as imperfect spheres, however, of different type. Namely, in the present paper, we consider the coefficient of restitution for the repeated bounce of smooth but slightly eccentric particles of dumbbell type. This is directly related to experiments where the coefficient of restitution was obtained from the time lap between consecutive impacts of a particle bouncing on a flat plate [12–17].

II. MODEL

The particle model considered here is sketched in Fig. 1.

The shape of the particle of homogeneous density is described by the surface of two identical spheres of radius R and the distance of their centers L with $0 \leqslant L \leqslant 2R$. The center of mass of the particle is $\vec{r}_G$.

When the particle is dropped from a certain height and bounces repeatedly against a smooth horizontal solid plane, the ratio between the normal components of the rebound velocity and the impact velocity defines the coefficient of restitution for the ith impact:

$$\varepsilon = -\frac{\vec{v}_G'(\tau_i)\cdot\vec{n}}{\vec{v}_G(\tau_i)\cdot\vec{n}}, \tag{1}$$

where $\vec{n}$ is the unit vector perpendicular to the plane. At time τ_i, the velocity of the center of mass of the particle $\vec{v}_G(\tau_i)$ (impact velocity) turns into the postcollisional velocity $\vec{v}_G'(\tau_i)$ (rebound velocity) due to the collision which is assumed an instantaneous event. For spherical particles, ε depends on the impact velocity at the point of contact (linear and angular) and material properties. For nonspherical particles, ε depends, moreover, sensitively on the angular orientation of the particle at the instant of the collision. Considering the latter quantity as random, the coefficient of restitution becomes a fluctuating quantity to be characterized below.

Consider the motion of a particle starting after the ith impact. The dynamics of the particle consists of alternating modes of (a) free flight for time intervals $t \in (\tau_i, \tau_{i+1})$, $i = 1, 2, \ldots$ where the only force acting on the particle is gravity and (b) collisions at times τ_i where the linear and angular momenta of the particles change instantaneously. Let us describe both stages in detail.

Starting at τ_i, the next collision following i occurs when either of the constituting spheres collides with the floor, whatever occurs first. Thus, the next collision, $i+1$, occurs at

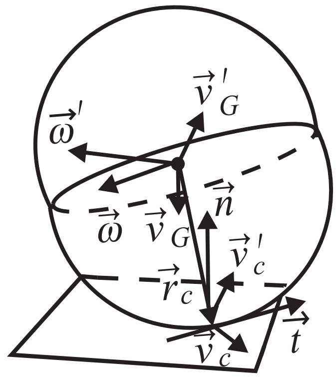

FIG. 1. Sketch of the collision of an aspherical particle with a plane at the instant of the collision. The particle consisting of two identical spheres of radius R and distance $L \ll R$ approaches the plane (normal unit vector $\vec{n}$) with pre-collisional center of mass velocity $\vec{v}_G$ and angular velocity $\vec{\omega}$. The corresponding post-collisional velocities are $\vec{v}_G'$ and $\vec{\omega}'$. The center of mass at position $\vec{r}_G$ is shown by a fat point. The velocity at the contact point at position $\vec{r}_c$ changes correspondingly from $\vec{v}_c$ to $\vec{v}_c'$. The directions of the projection of both velocities to the plane is given by the tangential unit vector $\vec{t}$.

time $\tau_{i+1} = \min(\tau_{i+1}^{(1)}, \tau_{i+1}^{(2)}) > \tau_i$, with

$$\left[\vec{r}_G\left(\tau_{i+1}^{(1)}\right) + \hat{A}\left(\tau_{i+1}^{(1)}\right)\vec{r}\,\right]_z = R,$$
$$\left[\vec{r}_G\left(\tau_{i+1}^{(2)}\right) - \hat{A}\left(\tau_{i+1}^{(2)}\right)\vec{r}\,\right]_z = R. \quad (2)$$

Here, $[.]_z$ denotes the vertical component, $\vec{r} \equiv (0,0,\frac{L}{2})$ and $-\vec{r}$ are the positions of the spheres in the body-fixed frame of reference, and $\hat{A}(t)$ describes the angular orientation of the particle by the angle ωt around the axis $\vec{e}_\omega \equiv \vec{\omega}(t)|\vec{\omega}(t)|$ [18]:

$$\hat{A}(t) = \hat{I}\cos\omega t + \vec{e}_\omega \otimes \vec{e}_\omega[1 - \cos\omega t] + \hat{\Omega}\sin\omega t. \quad (3)$$

Here, $\hat{I}$ is the unit matrix, $\otimes$ denotes the outer product, and

$$\hat{\Omega} \equiv \begin{pmatrix} 0 & -e_{\omega z} & e_{\omega y} \\ e_{\omega z} & 0 & -e_{\omega x} \\ -e_{\omega y} & e_{\omega x} & 0 \end{pmatrix} \quad (4)$$

is the skew symmetric form of unit vector $\vec{e}_\omega \equiv (e_{\omega x}, e_{\omega y}, e_{\omega z})$.

During the free flight between successive collisions, the vector of angular velocity, $\vec{\omega}$, is constant and the velocity of the center of mass changes only due to gravity, $\vec{g}$:

$$\vec{r}_G(\tau_{i+1}) = \vec{r}_G(\tau_i) + \vec{v}_G(\tau_i)(\tau_{i+1} - \tau_i) + \frac{\vec{g}}{2}(\tau_{i+1} - \tau_i)^2. \quad (5)$$

While the motion of the particle in between successive collisions is pure kinematics, described by Eqs. (2) and (5), at the time of the collision, τ_i, the linear velocity and angular velocity change instantaneously due to the linear and angular impulses, $\vec{P}$ and $\vec{r}_c \times \vec{P}$, where $\vec{r}_c$ is the vector from the center of mass to the point of contact between the particle and the floor,

$$\vec{r}_c = \hat{A}(\tau_i)\vec{r} - \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix}. \quad (6)$$

The impact velocity at the point of contact is given by

$$\vec{v}_c = \vec{v}_G + \vec{\omega} \times \vec{r}_c. \quad (7)$$

Writing the components of this vector in the direction of the plane normal, $\vec{n}$, and perpendicular to it,

$$\begin{aligned}\vec{v}_c &= [\vec{v}_c \cdot \vec{n}]\vec{n} + [|\vec{v}_c - (\vec{v}_c \cdot \vec{n})\vec{n}|]\left[\frac{\vec{v}_c - (\vec{v}_c \cdot \vec{n})\vec{n}}{|\vec{v}_c - (\vec{v}_c \cdot \vec{n})\vec{n}|}\right] \\ &= v_c^n \vec{n} + v_c^t \vec{t},\end{aligned} \quad (8)$$

where the expressions in square brackets define the components of the impact velocity, v_c^n and v_c^t, and the unit vector, $\vec{t}$, of the impact velocity perpendicular to $\vec{n}$ (in the plane), respectively.

The mechanics of the collision is fully determined by the change of the velocity at the point of contact due to the collision which may be characterized by means of the *microscopic* coefficients of normal and tangential restitution, ϵ^n and ϵ^t:

$$\begin{aligned} v_c^{n\prime} &= -\epsilon^n v_c^n, \\ v_c^{t\prime} &= \epsilon^t v_c^t, \end{aligned} \quad (9)$$

This change of the velocity at the contact point may be expressed in the form of a certain linear impulse, $\vec{P}$, and angular impulse, $\vec{r}_c \times \vec{P}$, being transferred to the particle at the instant of the collision:

$$-(\epsilon^n + 1)v_c^n\vec{n} + (\epsilon^t - 1)v_c^t\vec{t} = \frac{\vec{P}}{m} + \hat{J}^{-1}(\vec{r}_c \times \vec{P}) \times \vec{r}_c. \quad (10)$$

The mass, m, and moment of inertia tensor, $\hat{J}$, depend on the eccentricity, L, of the dumbbell particle. The derivation of both quantities can be found in the Appendix.

The postcollisional values of the center-of-mass velocity, $\vec{v}_G'$, needed for the computation of ε [see Eq. (1)] and the angular velocity, $\vec{\omega}'$, can be obtained from $\vec{P}$ via

$$\begin{aligned} \vec{v}_G' - \vec{v}_G &= \frac{1}{m}\vec{P}, \\ \vec{\omega}' - \vec{\omega} &= \hat{J}^{-1}(\vec{r}_c \times \vec{P}). \end{aligned} \quad (11)$$

III. NUMERICAL EXPERIMENT

By means of the previous expressions, we performed a numerical experiment corresponding to a bouncing-ball experiment. To this end, we perform the following steps:

(1) Release a particle from a certain height of its center of mass at random angular orientation and angular velocity $\vec{\omega} = \vec{0}$.

(2) Compute the time of the next impact via Eq. (2) using Eqs. (3) and (4).

(3) Compute the contact point, $\vec{r}_c$, and its velocity, $\vec{v}_c$, by means of Eqs. (6) and (7).

(4) Solve Eq. (10) for the linear impulse $\vec{P}$ and compute the angular impulse $\vec{r}_c \times \vec{P}$. The mass, m, and moment of inertia tensor, $\hat{J}$, of the particle are given by Eqs. (A3) and (A1) with Eqs. (A5) and (A8) (see the Appendix).

(5) Compute the postcollisional velocity of the center of mass, $\vec{v}_G'$, and the post-collisional angular velocity, $\vec{\omega}'$, using Eq. (11).

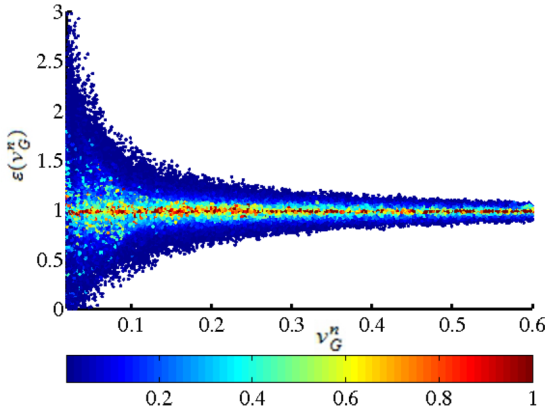

FIG. 2. (Color online) The macroscopic coefficient of restitution, ε, versus the impact velocity, v_G^n, for eccentricity $L/R = 0.1$. The graph shows an abundance of 400 000 impacts. The data points are colored according to the normalized frequency of occurrence (see text for explanation).

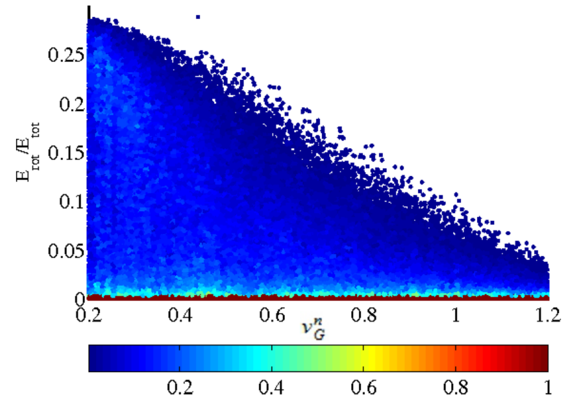

FIG. 3. (Color online) Ratio of rotational versus total energy, Eq. (12), as a function of the impact velocity, v_G^n. The data are colored according to the normalized frequency of occurrence.

(6) Evaluate Eq. (1) to obtain the (macroscopic) coefficient of restitution, ε.

(7) Exit when a predetermined number of bounces has been performed, otherwise continue with step III.

Figure 2 shows the measured coefficient of restitution, ε, as a function of the impact velocity $v_G^n \equiv -\vec{v}_G \cdot \vec{n}$. System parameters are asphericity $L/R = 0.1$ and microscopic coefficient of restitution, $\epsilon_n = 0.98$, $\epsilon_t = 1$ (no energy dissipation in tangential direction). Each of the 400 000 data points stands for the (macroscopic) coefficient of restitution, ε, obtained in the simulation. The color codes are for the frequency of occurrence of certain data points of (v_G^n,ε). The color of a data point was determined in the following way: We define a lattice with origin $(v_G^n,\varepsilon) = (0,0)$ and mesh size $\Delta v_G^n = 0.002$, $\Delta\varepsilon = 0.01$ and count the number of data points populating each lattice site. The color code gives the population of a site normalized by the largest population value.

From Fig. 2 we notice that the coefficient of restitution may exceed unity, which seems to violate conservation of energy. This effect is due to the exchanges between translational and rotational energy. Since only the energy of the translational motion enters the coefficient of restitution, Eq. (1), the energy of the rotational degrees of freedom may be considered as an internal reservoir of energy of the particle which couples to the translational degrees of freedom. This way, the postcollisional linear velocity perpendicular to the plane may be larger than the precollisional value, at the expense of rotation. The exchange of energy is, therefore, the origin of the fluctuations shown in Fig. 2.

Looking to the scatter of the data shown in Fig. 2 we find an increasing uncertainty of the data (enhanced scatter) with decreasing impact velocity, that is, with increasing number of impacts. The same effect is observed in experiments [10] and simulations of other nonspherical rough particles [11]. Similarly as superelastic collisions, also the widening of the distribution with increasing number of bounces may be explained by the coupling between rotation and translation: At the time of the first impact, the particle does not have any rotational energy but a large amount of translational energy. Therefore at the first impact, the particle can only gain but not lose rotational energy. Consequently, at this stage, the rotational degrees of freedom become excited by transferring energy from the translational motion to the rotational degrees of freedom. This leads to an extra loss of kinetic energy of the translational motion, thus, to a macroscopic (measured) coefficient of restitution $\varepsilon < \epsilon_n$. At later stages when the reservoir is charged, the rotational degrees of freedom may transform back to translational motion leading to $\varepsilon > \epsilon_n$. From this argument we see immediately that the absolute of fluctuations, $|\varepsilon - \epsilon|$, is intimately related to the amount of rotational energy of the particle which increases on average with the number of bounces. Figure 3 shows the ratio of the rotational versus total energy,

$$\frac{E_{\mathrm{rot}}}{E_{\mathrm{tot}}} = \frac{E_{\mathrm{rot}}}{E_{\mathrm{rot}} + E_{\mathrm{tran}}} = \frac{\vec{\omega} \cdot \hat{J}\vec{\omega}}{\vec{\omega} \cdot \hat{J}\vec{\omega} + m\vec{v}_G^{\,2}}. \tag{12}$$

Although the ratio $E_{\mathrm{rot}}/(E_{\mathrm{rot}} + E_{\mathrm{tran}})$ increases monotonously in Fig. 3, we wish to point out that the transfer to and from the reservoir is not trivial but depends in a complicated way on the geometry of the impact. In particular, it is not a purely cumulative effect; this can be seen, e.g., from the fact that asymptotically both components of the energy, translation and rotation, will cease.

For a more quantitative presentation of the probability density than the color coding shown in Fig. 2, we analyze the frequency of occurrence of certain values of ε for small intervals of the velocity separately (Fig. 4). Here and in the following, the notation $p_X(s)$ stands for the probability density of the random variable X at the argument s. Similarly, $P(X \leqslant s)$ stands for the cumulative probability distribution.

Similar as in experiments [10] we obtain a probability density composed of two exponential functions (asymmetric Laplace function) which may be explained for rough spheres by detailed stochastic analysis of the impact geometry [11]. For very small velocity, the probability density deviates from the Laplacian (data for $\vec{v}_G^n = 0.1$ m/s in Fig. 4). For such small velocities, the angular orientations of the particle at times of successive impact are not statistically independent. Instead, the height of the jumps is much smaller than R such that numerical experiments show that the particle jumps from

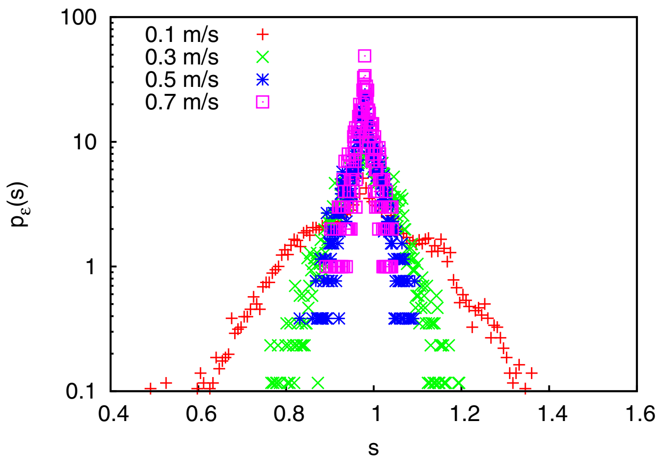

FIG. 4. (Color online) Histograms of ε for four different impact velocities.

one sphere of the dumbbell to the other and, consequently, the linear impulses, $\vec{P}$, and angular impulses, $\vec{r}_c \times \vec{P}$, are strongly correlated. One of the preconditions in [11] leading to the Laplacian shape was the statistical independence of the collisions. Therefore, the deviation from the Laplacian for very small velocity may be attributed to correlations of the impacts. We believe that for most bouncing-ball experiments reported in the literature, this regime is not relevant; however, there are experiments reported by King *et al.* [14] where impact velocities as low as 0.001 m/s are considered such that the described motion may become relevant. Analyzing the probability density of the ratio $E_{\text{rot}}/(E_{\text{rot}} + E_{\text{tran}})$ in the same way (Fig. 5), we obtain decaying functions whose slope decreases with decreasing impact velocity (and such increasing number of bounces). This observation is consistent with the accumulation of rotational energy explained above. The result is, moreover, consistent with the result for rough spheres [11].

Another characteristic to quantify the fluctuations of the coefficient of restitution is the standard deviation σ as a function of the normal component of the impact velocity. Figure 6 shows the standard deviation for dumbbells of

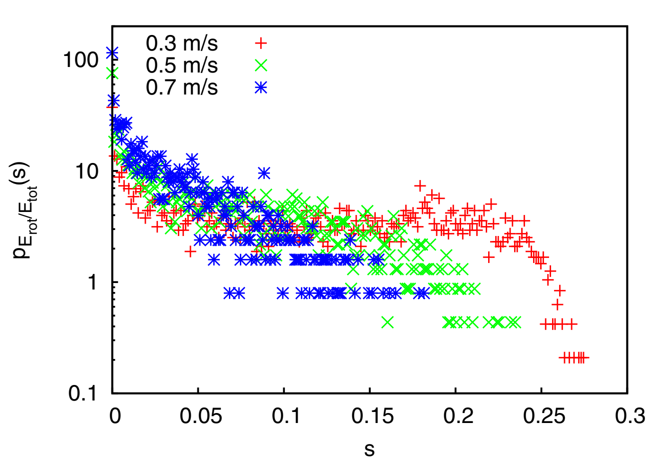

FIG. 5. (Color online) Histograms of $E_{\text{rot}}/E_{\text{tot}}$ for four different impact velocities.

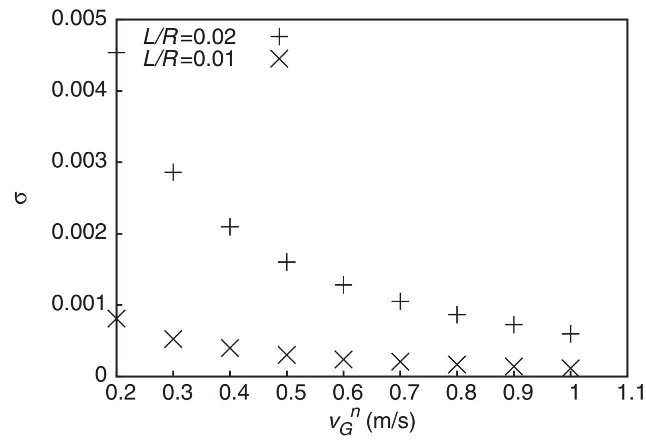

FIG. 6. Standard deviation of the fluctuating coefficient of restitution as a function of the normal component of the impact velocity. The eccentricity, $L/R = 0.01$ and $L/R = 0.02$, corresponds to average quality glass spheres [19].

very low eccentricity, $L/R = 0.01$ and $L/R = 0.02$, which is typical for glass spheres [19] as frequently used in bouncing-ball experiments. Although the error is rather small for small deviations from the spherical shape it can be shown [20] that for bouncing-ball experiments, the error originating from the eccentricity dominates the total error of the experiment and cannot be disregarded.

IV. NEGATIVE COEFFICIENT OF RESTITUTION

When we extend in Fig. 2 the range of impact velocity to lower values, besides values $\varepsilon > 1$ we obtain values $\varepsilon < 0$ (see Fig. 7).

At first glance, negative values of the coefficient of restitution may look rather academic; however, for realistic particles these values appear quite naturally. This effect was first reported by Saitoh *et al.* [21] for simulations of adhesive colliding clusters of nanoparticles at large impact velocity.

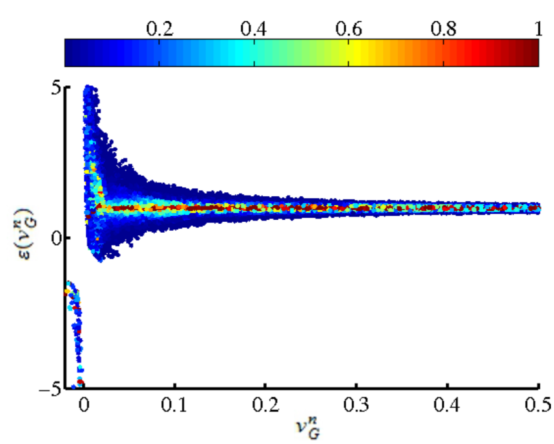

FIG. 7. (Color online) Same data as Fig. 2 but for smaller values of the argument. We observe $\varepsilon < 0$ for low impact velocity. There are two different mechanisms leading to negative values of the coefficient of restitution corresponding to the two groups of data points appearing in the figure (for discussion see the text).

Later, it was shown analytically that the phenomenon of negative coefficients of restitution is a much more general phenomenon [22] which occurs for *all* realistic particles independent of the concrete details of particle interaction, only due to finite duration of collisions. Clearly, every collision of realistic particles, will last for some time period, that is, the model of instantaneous interaction is an idealization.

For the case of dumbbell particles, however, the mechanisms leading to negative values of the coefficient of restitution are different from those discussed in [21,22]: First, a negative precollisional value of the center-of-mass velocity (fall due to gravity) may lead to a negative postcollisional velocity due to rotation of the particle, that is, the center of mass continues approaching the plane after the impact resulting in a negative value of the coefficient of restitution as defined by Eq. (1). Obviously, the probability of such events increases with increasing impact number when the rotational degree of freedom gained sufficient energy such that $|\omega \times \vec{r}_c|$ is not negligible compared to $|\vec{v}_G|$. This mechanism explains the negative values in the right branch ($v_G^n > 0$) in Fig. 7. As a second mechanism, a rotating dumbbell particle can contact the plane even if its center-of-mass velocity departs from the plane. Here a positive precollisional velocity leads to a positive postcollisional velocity and, consequently, to a negative coefficient of restitution. The second mechanism explains the negative values in the left branch ($v_G^n < 0$) in Fig. 7.

Certainly, the probability to obtain a value $\varepsilon < 0$ increases with increasing rotational energy (see Fig. 7 and arguments above). Thus one might come to the conclusion that the appearance of negative coefficients of restitution is a purely cumulative effect. However, we will show below that even for $|\vec{\omega}| = 0$, there is a finite probability to obtain negative values. This case allows for an analytical description: Using Eqs. (1) and (11), we write

$$\varepsilon = -1 - \frac{1}{m}\frac{\vec{P}\cdot\vec{n}}{\vec{v}_G\cdot\vec{n}}. \tag{13}$$

The impulse $\vec{P}$ is found by solving the system, Eq. (10), with the components of $\vec{r}_c = (r_{cx}, r_{cy}, r_{cz})$ and $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$, the inertia tensor $\hat{J}$ given by Eqs. (A1), (A5), and (A8) in the Appendix and assuming $\epsilon^t = 1$:

$$\vec{P} = m(1+\epsilon_n)\frac{(\vec{v}_G\cdot\vec{n} + \omega_x r_{cy} - \omega_y r_{cx})}{\vec{r}_c^{\,2} + J_{xx}/m}\begin{pmatrix} r_{cx}\, r_{cz} \\ r_{cy}\, r_{cz} \\ J_{xx}/m + r_{cz}^2 \end{pmatrix}. \tag{14}$$

Inserting in Eq. (13) yields

$$\varepsilon = -1 + \frac{1+\epsilon_n}{\vec{v}_G\cdot\vec{n}} \times \frac{(\vec{v}_G\cdot\vec{n} + \omega_x r_{cy} - \omega_y r_{cx})\left(J_{xx}/m + r_{cz}^2\right)}{\vec{r}_c^{\,2} + J_{xx}/m}. \tag{15}$$

Equation (15) provides the relation between the macroscopic coefficient of restitution as defined in Eq. (1) and the microscopic coefficient of restitution, ϵ^n, describing the ratio of pre- and postcollisional velocity at the point of contact. The vector of the point of contact can be expressed by the angles θ and ϕ and the distance between the two spheres L (see Fig. 8).

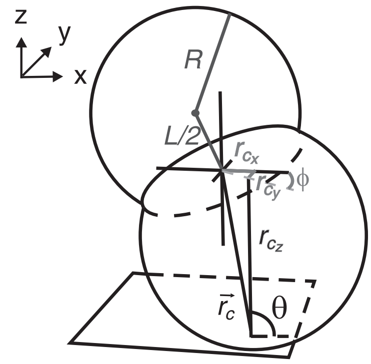

FIG. 8. Definition of variables.

$$\vec{r}_c = \left(\frac{\sin\theta\,\cos\phi L}{2}, \frac{\sin\theta\,\sin\phi L}{2}, \frac{-\cos\theta}{2}L - R\right). \tag{16}$$

With $\vec{\omega} = (0,0,0)$ Eq. (15) adopts the form

$$\varepsilon = -1 + \frac{(1+\epsilon_n)(4J_{xx}/m + L^2\cos^2\theta + 4LR\,\cos\theta + 4R^2)}{4J_{xx}/m + 4LR\,\cos\theta + 4R^2 + L^2}. \tag{17}$$

For vanishing rotational velocity, the problem reduces to two dimensions such that the coefficient of restitution ε is described by the angle θ between the symmetry axis of the particle and the z axis (Fig. 8).

We assume that the angle θ is homogeneously distributed in the interval $\theta \in [0,\pi)$. Consequently, the cumulative probabilities of $\cos\theta$ read

$$\begin{aligned} P(\cos\theta \leqslant s) &= 1 - \{P\,(\theta \in [0, \arccos s]) \\ &\quad + P(\theta \in [\pi - \arccos s, \pi])\} \\ &= 1 - \frac{1}{\pi}\arccos(s). \end{aligned} \tag{18}$$

Differentiating Eq. (18) we obtain the probability density:

$$p_{\cos(\theta)}(s) = \frac{1}{\pi}\frac{1}{\sqrt{1-s^2}}. \tag{19}$$

Equation (17) contains only a single fluctuating quantity, which is $\cos(\theta)$. Hence, the distribution of $\varepsilon = h(\cos\theta)$ where h is a function of $\cos(\theta)$ corresponding to Eq. (17) can be calculated by the transformation

$$p_\varepsilon(s) = p_{\cos(\theta)}[h^{-1}(s)]\frac{dh^{-1}(s)}{d\varepsilon}. \tag{20}$$

We obtain

$$p_\varepsilon(s) = \frac{2Rf(s) + \left[\frac{1}{2}k(4J_{xx}/m + L^2) + 4R^2(2s - \epsilon_n + 1)\right]}{\pi f(s) kL\sqrt{1 - \left(\frac{2R(s-\epsilon_n)+f(s)}{kL}\right)^2}}, \tag{21}$$

$$f(s) = \sqrt{4(s-\epsilon_n)[(s+1)R^2 + kJ_{xx}/m] + k(s+1)L^2}, \tag{22}$$

where $k = (1+\epsilon_n)$.

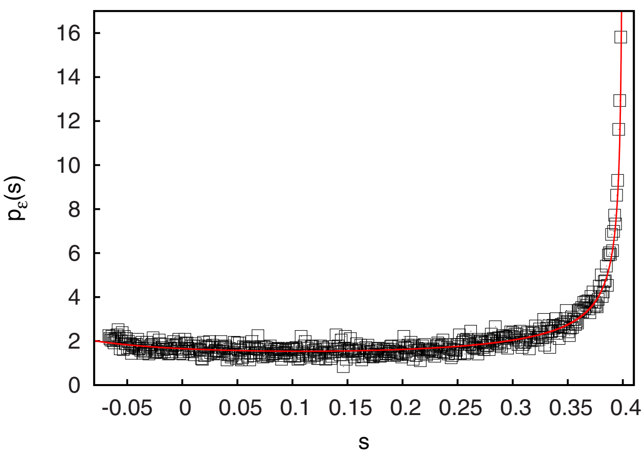

FIG. 9. (Color online) Probability density of the coefficient of restitution as found in simulations together with the analytical result.

A comparison with a simulation for $\epsilon_n = 0.4$ and $L = 2R$ shows almost perfect agreement (see Fig. 9). We observe negative values of the coefficient of restitution.

V. CONCLUSION

The coefficient of restitution is an important quantity characterizing the collision of dissipative particles and, thus, to describe the dynamic properties of dilute to moderately dense granular systems. It is the foundation of both kinetic theory of granular gases and rapid granular flows as well as of event-driven molecular dynamics of dilute granular systems. So far, almost all kinetic theory and event-driven molecular dynamics simulations assume that the particles are perfect spheres such that the coefficient of restitution is a deterministic function of material properties, radii and impact velocity. One way to extend the kinetic theory to nonspherical particles is to consider the angular orientation of the particles at the instance of the collision as random variables which may be justified for uncorrelated collisions being a necessary condition for the practical application of kinetic theory anyway.

The assumption of a deterministic function for the coefficient of restitution based on the perfectly spherical shape of the particles is not only a strong idealization for ordinary granular systems but it is even questionable for particles which are assumed to be spherical such as polished glass beads or ball-bearing balls. This can be seen in many experimental results published in the literature, e.g., [12–17], where the strong scatter of the measured coefficient of restitutions cannot be explained by the imperfections of the experiment [10]. This scatter can be attributed to microscopic imperfections of the surface of the particles [11] in very good agreement with the experiment [10].

While in [10,11] a particle is described by a central sphere with a large number (typically 10^6 or more) of asperities intended to characterize some surface roughness, in the present paper we follow a complementary approach. Namely, we assume that the particles are smooth but of slightly eccentric shape.

Our results show that even for rather small eccentricity, the coefficient of restitution reveals significant fluctuations originating from the energy transfer between the translational and rotational degrees of freedom. The substantial increase of the data scattering with the decrease of impact velocity is attributed to the growing role of the rotational degrees of freedom in the energy partition. Similar as reported in [11], the probability density of the coefficient of restitution has a Laplacian form, in agreement with experiments [10]. The obtained probability density implies a certain finite probability for negative values of the coefficient of restitution. This effect is not purely cumulative due to the rotation of the particle but applies also for collisions of particles with vanishing precollisional rotation velocity. Negative values of the coefficient of restitution appear if either the precollisional normal relative velocity is negative (that is, the center of mass approaches the plane) and the postcollisional velocity is negative as well, or if both pre- and postcollisional velocities are positive. Both situations may be understood from the geometry of a rotating dumbbell.

From our results we conclude that the assumption of a fluctuating coefficient of restitution is a possible way to extend the kinetic theory of granular gases to slightly nonspherical particles. Since rotational and translational degrees of freedom are coupled in a nontrivial way [23,24], we expect substantial differences of the kinetic properties of such gases as compared to gases of spherical particles. We conclude also that the strong scatter in bouncing-ball experiments measuring the coefficient of restitution may be attributed to both surface roughness [11] and eccentricity as both types of imperfection lead to the same probability density of the scatter of Laplace type, also observed in experiments. In reality, we believe the scatter originates from both types of imperfection. Finally, we believe that the eccentricity should be taken into account when estimating the error in bouncing-ball measurements of the coefficient of restitution. We believe that the deviation of the shape of the particle from the perfect sphere is one of the largest sources of uncertainty and may even dominate the total error of the measurement [20].

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APPENDIX: MASS AND MOMENT OF INERTIA TENSOR OF A DUMBBELL

The shape of the dumbbell is given by the volume occupied by two spheres of radius R and distance L. Homogeneous density, ρ, is assumed. For the computation of the mass, m, and the moment of inertia tensor, $\hat{J}$, we assume that the spheres are located at $\vec{r} = (0,0,\frac{L}{2})$ and $-\vec{r}$ such that the principle axes coincide with the axes of the coordinate system and the moment of inertia tensor assumes the form

$$\hat{J} = \begin{pmatrix} J_{xx} & 0 & 0 \\ 0 & J_{xx} & 0 \\ 0 & 0 & J_{zz} \end{pmatrix}, \tag{A1}$$

where the symmetry $J_{xx} = J_{yy}$ was taken into account. We restrict ourselves to the case $L \leqslant 2R$, which is sufficient for the dumbbell particles of low eccentricity considered here. The generalization to arbitrary $L \geqslant 0$ is straightforward.

For the computation of the mass and moment of inertia we exploit the symmetry of the body with respect to the $x-y$ plane. The part of the body in the half space $z \geqslant 0$ is limited by the function

$$\varrho^2(z) = R^2 - \left(z - \frac{L}{2}\right)^2, \quad 0 \leqslant z \leqslant R + \frac{L}{2}. \tag{A2}$$

Therefore, the mass of the dumbbell particle is

$$m = 2\rho\pi \int_0^{R+(L/2)} \varrho^2(z)\, dz = \rho\pi\left(\frac{4}{3}R^3 + R^2L - \frac{1}{12}L^3\right). \tag{A3}$$

The (infinitesimal) moment of inertia of a cylinder of radius ϱ and height dz rotating around its axis (wheel mode) and around its diameter (flipping coin mode) are

$$dJ_c^{\text{wheel}} = \frac{\pi\rho}{2}\varrho^4 dz, \quad dJ_c^{\text{coin}} = \frac{\pi\rho}{4}\varrho^4 dz. \tag{A4}$$

From the first expression we obtain the moment of inertia of the dumbbell with respect to rotation around the z axis:

$$\begin{aligned} J_{zz} &= 2\int_0^{R+(L/2)} J_c(z) dz = 2\frac{\pi\rho}{2}\int_0^{R+(L/2)} \varrho^4(z) dz \\ &= \pi\rho\left(\frac{8}{15}R^5 + \frac{1}{2}R^4L - \frac{1}{12}R^2L^3 + \tfrac{1}{180}L^5\right), \end{aligned} \tag{A5}$$

where Eq. (A2) was used. The other element of the moment of inertia tensor, J_{xx}, reads

$$J_{xx} = 2\int_0^{R+(L/2)} J_c(z) dz = 2\rho\pi \int_0^{R+(L/2)} \left(\frac{1}{4}\varrho^4 + z^2\varrho^2\right) dz, \tag{A6}$$

where the first term in the last integrand corresponds to dJ_c^{coin} given in Eq. (A4) and the second term appears due to the shift of the rotation axis by z. Using again Eq. (A2) for $\varrho(z)$, we obtain

$$\begin{aligned} J_{xx} &= \rho\pi\int_0^{R+(L/2)} \left(\frac{1}{2}R^2 - \frac{1}{8}L^2\right)^2 + \left(2LR^2 - \frac{1}{2}L^3\right)z \\ &\quad + \left(2R^2 + \frac{1}{2}L^2\right)z^2 + 2Lz^3 - 3z^4\, dz, \end{aligned} \tag{A7}$$

$$J_{xx} = \rho\pi\left(\frac{8}{15}R^5 + \frac{3}{4}R^4L + \frac{1}{3}R^3L^2 + \frac{1}{24}R^2L^3 + \frac{1}{960}L^5\right). \tag{A8}$$

The tensor elements agree with the alternative derivation given in [25]. To summarize, the mass of the dumbbell characterized by R, L, and ρ is given by Eq. (A3). Its moment of inertia tensor in the system of principal axes is given by Eq. (A1) with Eqs. (A5) and (A8). The inertia tensor in the rotated coordinate system is $\hat{A}(t)\hat{J}\hat{A}(t)^{-1}$.

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