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A simple geometrical model for solid friction

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We present a simple model for the friction of two solid bodies moving against each other. In a self-consistent way we can obtain the dependence of the macroscopic friction force as a function of the driving velocity, the normal force and the ruggedness of the surfaces in contact. Our results are discussed in the context of friction laws used in earthquake models.

The friction of two solid bodies against each other has been studied for many centuries. Coulomb [1] formulated the law that the friction force F_{fr} slowing down the relative motion between the two bodies is proportional to the normal force F_n and the proportionality constant is called the friction coefficient μ. In particular, within this law, the friction force is supposed to be independent on the area of contact and on the relative velocity v with which the two bodies move against each other. Experimentally it is, however, well known [2,3] that F_{fr} does depend on v.

A more detailed knowledge of the dependence of μ on the shearing velocity v is very important in earthquake models. Using a function of the form

$$\mu(v) = \beta/(v + v_0)\ , \tag{1a}$$

it has been shown [4] that the classical block-spring model of Burridge and Knopoff [5] is able to reproduce realistic events including the Gutenberg–Richter law for the distribution of earthquake sizes. A friction law of the form

$$\mu(v) = \mu_0 + \beta \ln\left(\frac{v}{v_0}\right) \tag{1b}$$

has been introduced [6] and used to describe slip motion of faults [7]. The

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question arises: Is it possible to obtain the friction laws of eqs. (1) also from a microscopic model?

To that purpose one has to know the microscopic origin of friction. Already in 1699 Amontons and de la Hire [3] proposed that the main contribution to solid friction just comes from the geometrical ruggedness of the two surfaces in contact. Since the two surfaces are not perfectly flat they cannot move at the same time straight and maintain perfect contact. The surfaces must separate a distance at least as far apart as their highest asperities and they will jump up and down. This vertical jumping will be attenuated by the elasticity of the bulk but on a local level the surfaces will move apart when asperities hinder each other and come close when their indentations happen to fit better into each other.

Of course if one goes into detail many more physical effects come into play [2,3]. Enormous forces act at the contact of the tips of asperities so that these tips are plastically deformed and can even fragment off (wear). The fragments and eventual intermediate fluids can lubrify the contacts, i.e. form layers that smoothen the surfaces. The fragments can even exert the role of bearings [8]. Also on the molecular level adhesion or electrostatic forces can play an important role.

Unfortunately, however, even the simplest problem, namely the purely geometrical hindrance of the two surfaces sliding against each other is already very difficult to deal with and from the theoretical point of view very little is yet known. It is the purpose of this work to shed more light on its understanding. In fact, this simple situation can be realized by considering infinitely strong and stiff elastic bodies in vacuum. Experimentally a good material for that purpose would be rocks.

In the last years the roughness of solid surfaces has attracted major interest mainly due to the introduction of the concept of scaling and self-affinity [9]. It has been verified experimentally [10] that the height variations z of rock and metal surfaces are invariant within a certain range under a scale transformation $z \rightarrow \lambda^{\zeta} z$ where the roughness exponent ζ is quite universally around 0.85. This scale invariance is the origin of anomalous elastic behaviour [11].

Geometrically the scale invariance means that one has asperities on all length scales. In particular one can decompose the topography of the surface into different scales of resolution and then expect that the behaviour be the same on each scale. We will use this idea in our approach to write down a closed expression for the friction as a function of the driving velocity for a simple model surface. Our principal assumption will be that the friction parameter μ is the same on all levels. This is a kind of mean field assumption which does not take into account the scaling exponent ζ. In fact, one can also think of formulating a renormalization procedure that describes how to go from one scale to the next using ζ and this work is in progress [12].

In our model we make several simplifying assumptions concerning the geometry of the surface. First we reduce the problem from three to two dimensions as if the surfaces were made out of grooves so that considering a cross section would be enough. Furthermore we suppose that all the grooves on one scale have the same size and spacing, and that they are identical for both lids. A schematic plot is shown in fig. 1. We consider asperities that just have a triangular shape of an inclination angle α and width B.

Gravity g pushes the two bodies together while they are sheared against each other with a fixed relative velocity v. We suppose that at the areas where the two surfaces are in contact the Coulomb law holds so that the force acting against the direction of the relative motion is proportional to the normal force with proportionality constant μ. Therefore this friction force has a magnitude

$$f_{\mathrm{fr}} = m\mu g \cos(\alpha) \pm F\mu \sin(\alpha) , \tag{2}$$

where the second term is due to the component of the force F pulling the two surfaces against each other. The force of eq. (2) is independent on the contact area which therefore drops out entirely of the problem and as a consequence the size of the considered system becomes irrelevant in our calculation.

When the bodies now move on top of each other they will jump up and down. When they move up their surfaces slide against each other as shown in fig. 1 (phase I of the motion) until the tips of the triangles exactly touch. After that they will fly down on a parabolic trajectory (phase II of the motion) until the surfaces hit again against each other. They can hit either on the left side (case a) or on the right side (case b) of the triangles on the lower surface. In the calculation all the three cases (I, IIa and IIb) must be considered separately. Moreover, one has to take into account that the upper lid must not necessarily hit the lower one within the next period B but can jump over N ($N \geqslant 1$) periods (fig. 2). Only in phase I a restoring force acts which can be calculated to be

$$F(\mu) = mg \frac{\sin(\alpha) + \mu \cos(\alpha)}{\cos(\alpha) - \mu \sin(\alpha)} . \tag{3}$$

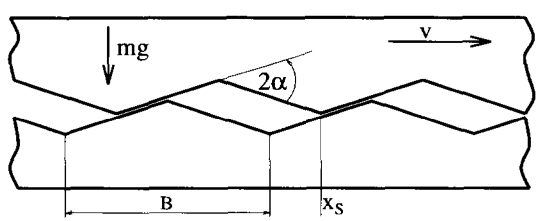

Fig. 1. Schematic plot of our model.

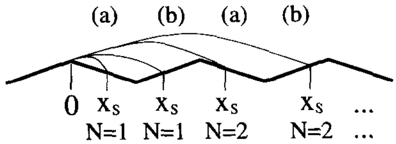

Fig. 2. Phases of the motion: in phases (a) and (b) the trajectories end on the right and left sides of the triangles respectively after jumping over N triangles.

In order to know how much time this force acts one must calculate the point x_s at which the surfaces hit against each other. These are in case a

$$x_s = \tfrac{3}{2}B + v^2\varphi + v\sqrt{\varphi^2 v^2 - \varphi(N-1)B} \tag{4a}$$

and in case b

$$x_s = \tfrac{3}{2}B + v\sqrt{B\varphi N}\ , \tag{4b}$$

with

$$\varphi = \frac{2\tan(\alpha)}{g}\ .$$

According to our previous discussion we now assume that also the entire system fulfils the Coulomb relation (at fixed velocity) with the same friction constant μ, because in our picture, when looking with a microscope, the flat surfaces of the triangles are themselves made out of many small triangles following the principles of self-similarity. This gives us the self-consistency equation

$$f = \frac{1}{T}\int_0^T F(\mu, t)\,\mathrm{d}t = m\mu g\ , \tag{5}$$

where $T = NB/v$ is the period, i.e. the time to go once through phases I and II. Solving eq. (5), we get

$$\mu = \frac{1}{2\tan(\alpha)}\left(\frac{\frac{1}{2}+\kappa}{N} - 1\right) + \sqrt{\frac{1}{4\tan^2(\alpha)}\left(\frac{\frac{1}{2}+\kappa}{N} - 1\right)^2 + \left(\frac{\frac{1}{2}-\kappa}{N}\right)}\ , \tag{6a}$$

with

$$\kappa = N - \tfrac{1}{2} - \frac{v}{B}\left(\sqrt{\varphi^2 v^2 - \varphi(N-1)B} + \varphi v\right)$$

if

$$v \in \left[\sqrt{\frac{B}{\varphi}}\sqrt{N-1}\,,\quad \sqrt{\frac{B}{\varphi}}\frac{N-\frac{1}{2}}{\sqrt{N}}\right]$$

(case a) and

$$\mu = \frac{\lambda}{\sqrt{\tan(\alpha)}} + \sqrt{\frac{\lambda^2}{\tan(\alpha)} + \lambda 2\sqrt{\tan(\alpha)} - 1}\,, \qquad (6b)$$

with

$$\lambda = v\sqrt{\frac{1}{2BgN}}\,,$$

otherwise (case b).

In fig. 3 we show the μ of eq. (6) as a function of v for various values of α. We clearly see the two cases a and b: when the surfaces hit on the left (right) side of the triangle, μ increases (decreases) with v. In between there is the case at which one has jumps from tip to tip and in that case μ vanishes. This happens at the characteristic velocity of the system which for dimensional reasons is $\sqrt{Bg}$. When the angle increases the friction increases and the a–b sequences become shorter (the jumps become longer). For small velocities and small angles Coulomb's original proposition seems fulfilled: μ does not depend on v. However, for very small velocities μ falls off to zero. This effect which gets stronger with α is an artifact of our self-consistent approach and therefore only rather small angles α (less than 5°) are allowed.

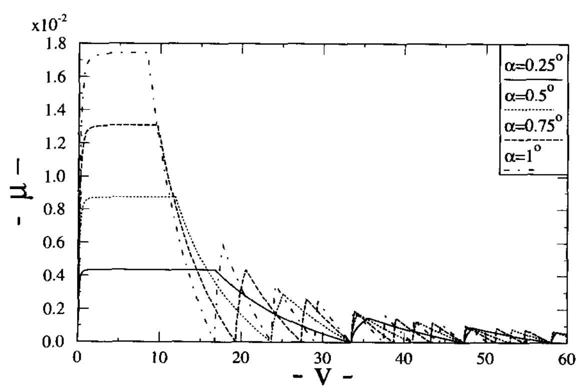

Fig. 3. Friction coefficient μ as a function of the driving velocity v for different angles α and $B = 0.0035/\tan(\alpha)$. We always use $g = 9.81$.

The zig zag behaviour of the curves in fig. 3 is due to the commensurability of the indentations of the two surfaces. In reality surfaces have random asperities and no synchronized motion as in our model is possible. In order to take care of this heterogeneity we now average the values of μ over an entire range of angles α changing B such that the height of the triangles remains the same. This can be seen in fig. 4 for the case in which we average over all α between 0.2° and 2.0° with the same probability for each angle. As expected one finds a smooth behaviour. In fig. 4 we have fitted these data to a curve of

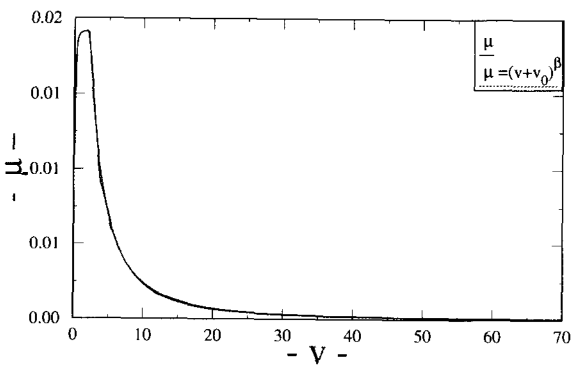

Fig. 4. Friction coefficient μ as a function of v, homogeneously averaged over α-values that range from 0.2° to 2.0°. Superposed we see the function $\mu = (v + v_0)^{\beta}$ with $v_0 = 3.308$ and $\beta = -2.342$ fitted for $2.2 \leq v \leq 70$.

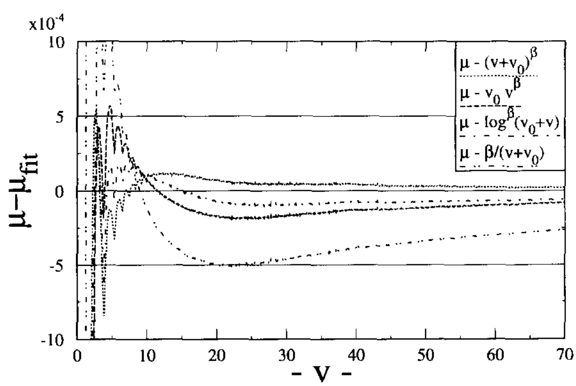

Fig. 5. Difference between our result for $\mu(v)$ and the best fits to various possible functional forms $\mathscr{F}(v)$ plotted as function of v, corresponding to $\mathscr{F}(v) = \log^{\beta}(v + v_0)$ with $v_0 = 4.431$ and $\beta = -6.183$; $\mathscr{F} = (v + v_0)^{\beta}$ with $v_0 = 3.308$ and $\beta = -2.342$; $\mathscr{F} = \beta/(v + v_0)$ with $v_0 = -1.198$ and $\beta = 0.0224$ and $\mathscr{F} = v_0 v^{\beta}$ with $v_0 = 0.0645$ and $\beta = -1.438$, fitted for $2.2 \leq v \leq 70$.

the form $\mu = (v + v_0)^{\beta}$ with $v_0 = 3.308$ and $\beta = -2.342$ and find very good agreement. We also tried several other possible functional forms for μ including the ones of eq. (1). In fig. 5 we see how well they can be described by our model by showing the difference between our values and the best fits to a large range of velocities. An ansatz due to eq. (1a) is fitted using $v_0 = -1.198$ and $\beta = 0.0224$ and gives reasonable agreement while the analogous fit using eq. (1b) shows deviations beyond the scale of fig. 5 and only works well in a smaller range of v. Other functional forms like $\log^{\beta}(v + v_0)$ and $v_0 v^{\beta}$ in fact fit very well too.

We have presented a very simple geometrical model in order to calculate the friction coefficient from the steric hindrances of the contacting surfaces. By assuming the same friction coefficient on different scales we can formulate a self-consistent equation. We take disorder into account by a very crude averaging procedure. The results of this approximate calculation are in qualitative agreement with the observed behaviour but clearly also other functional forms for the velocity dependence of μ work very well. In fact, we might expect that the various expressions would work equally well as assumptions for earthquake models.

Our purely geometrical approach has an analogy to the case of granular media in which traditionally a phenomenological static friction between the grains is introduced [13] in order to describe the finite angle of repose of a heap [14]. Recently, however, the insight was gained that the angles of repose can also be controlled just by varying the shape of the grains [15] in a model with no static friction. Therefore also for granular media the friction can be described only by the geometry of the asperities (grain shapes).

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