

Hydrodynamics of granular gases of viscoelastic particles

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Our study examines the long-time behaviour of a force-free granular gas of viscoelastic particles, for which the coefficient of restitution depends on the impact velocity, as it follows from the solution of the impact problem for viscoelastic spheres. Starting from the Boltzmann equation, we derived the hydrodynamic equations and obtained microscopic expressions for the transport coefficients in terms of the elastic and dissipative parameters of the particle material. We performed the stability analysis of the linearized set of equations and found that any inhomogeneities and vortices vanish after a long time and the system approaches the flow-free stage of homogeneous density. This behaviour is in contrast to that of a gas consisting of particles which interact via a (non-realistic) constant coefficient of restitution, for which inhomogeneities (clusters) and vortex patterns have been proven to arise and to continuously develop.

Keywords: granular gases; dissipative gases; hydrodynamics; inelastic collisions; clustering; structure formation

1. Introduction

For the theoretical description of granular materials as dissipative many-particle systems, it is frequently assumed that each particle (grain) consists of a macroscopic number of atoms but the system itself consists of a large number of grains. Insofar as such systems behave in many respects as a continuum medium, they may be, in principle, described by a set of hydrodynamic equations with appropriate boundary conditions. Although this approach is widely used in different fields of engineering and soil mechanics (see, for example, Kolymbas 2000; Vermeer *et al.* 2001), a first-principle theory is still found lacking and continuum mechanics of dense granular systems is still far from being understood.

For the much simpler case of rarefied granular systems, called ‘granular gases’ (Pöschel & Luding 2000), however, the apparatus of statistical mechanics which has been elaborated for molecular gases can be applied. With the necessary modifications one can derive a continuum description of such systems and investigate their large-scale dynamics.

The main difference between granular gases and molecular gases is the dissipative nature of particle collisions. Therefore, during the evolution of a homogeneously initialized force-free granular gas the average kinetic energy of the particles steadily decreases and may be described by the decay of the granular temperature. Cooling, however, is not the only process which characterizes the granular gas dynamics: during its evolution the initially homogeneous system becomes unstable and a rich variety of spatio-temporal structures develops, such as vortexes, e.g. a macroscopic flow fields and clusters, i.e. dense and rarefied regions. Many examples of structures in granular gases can be found in Pöschel & Luding (2000).

These structure formation processes have been observed in simulations and theoretically explained for granular gases, the collision of whose particles is described by a constant coefficient of restitution. The assumption $\varepsilon = \text{const.}$ simplifies the theoretical analysis substantially; however, it does not agree with experimental observations (see, for example, Bridges *et al.* 1984) and even contradicts basic mechanics (Ramírez *et al.* 1999). The coefficient of restitution ε depends noticeably on the impact velocity g , and tends to unity for very small g . As a consequence particles behave more and more elastically for decreasing impact velocity. The function $\varepsilon = \varepsilon(g)$ (for definitions see §2) has been derived for viscoelastic spheres (Brilliantov *et al.* 1996; Schwager & Pöschel 1998; Ramírez *et al.* 1999). It has been shown that the properties of a cooling granular gas during the first stage of its evolution alter qualitatively if $\varepsilon = \varepsilon(g)$ is applied (Brilliantov & Pöschel 2000*a, c*).

There arises the question as to whether structure formation processes such as clustering and vortex formation occur in granular gases of viscoelastic particles with $\varepsilon = \varepsilon(g)$. To answer this question we will derive hydrodynamic equations for such gases and study their stability. The paper follows the schedule below.

- (1) Discussion of the coefficient of restitution for colliding viscoelastic spheres.
- (2) Generalization of the Boltzmann equation with respect to the coefficient of restitution derived in §2 and analysis of the granular gas in the homogeneously cooling state, when the system is spatially uniform and no macroscopic flows are present.
- (3) Derivation of the hydrodynamic equations and the transport coefficients from the Boltzmann equation applying the Chapman–Enskog (or Grad) method.
- (4) Analysis of the hydrodynamic equations with respect to stability of the homogeneously cooling state.

2. The coefficient of restitution for viscoelastic spheres

The coefficient of (normal) restitution ε relates the normal components of the relative velocities before and after a collision, $\mathbf{v}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$ and $\mathbf{v}_{ij}^* \equiv \mathbf{v}_i^* - \mathbf{v}_j^*$ by $|\mathbf{v}_{ij}^* \cdot \mathbf{e}| = \varepsilon |\mathbf{v}_{ij} \cdot \mathbf{e}|$ ($g^* = \varepsilon g$ as a shorthand), i.e. it governs the microscopic dynamics of colliding particles. The unit vector $\mathbf{e} = \mathbf{r}_{ij}/|\mathbf{r}_{ij}|$ gives the direction of the intercentre vector $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ at the instant of the collision. From the conservation of momentum one finds the change of velocity for the colliding particles:

$$\mathbf{v}_i^* = \mathbf{v}_i - \frac{1}{2}(1 + \varepsilon)(\mathbf{v}_{ij} \cdot \mathbf{e})\mathbf{e}, \quad \mathbf{v}_j^* = \mathbf{v}_j + \frac{1}{2}(1 + \varepsilon)(\mathbf{v}_{ij} \cdot \mathbf{e})\mathbf{e}. \quad (2.1)$$

For elastic collision the coefficient of restitution equals unity, $\varepsilon = 1$; for inelastic collisions, $\varepsilon < 1$, it decreases with increasing inelasticity.

In the literature it is frequently assumed that the coefficient of restitution is a material constant, $\varepsilon = \text{const}$. Experiments, however, disprove this assumption: it was observed for colliding ice particles (Bridges *et al.* 1984) that ε increased by a factor of 2 when the impact velocity decreased by a factor of 10. Similar noticeable dependence of ε on the impact velocity was observed for other materials (Brach 1989; Wall *et al.* 1990; Kuwabara & Kono 1987). Very general analysis of the two-body contact problem also shows that none of the known interparticle forces (both elastic and inelastic) is compatible with the assumption $\varepsilon = \text{const}$. (Ramírez *et al.* 1999).

The simplest physically meaningful assumption for dissipative material behaviour is viscoelasticity, i.e. the elastic and dissipative stresses in the bulk of the particle material depend linearly on the strain and the strain rate, respectively (Landau & Lifshitz 1965). Generalizing Hertz's contact problem (Hertz 1882) with the assumption of viscoelasticity, we obtain the coefficient of restitution for colliding spheres (Brilliantov *et al.* 1996; Schwager & Pöschel 1998),

$$\varepsilon = 1 - C_1 A \alpha^{2/5} |\mathbf{e} \cdot \mathbf{v}_{ij}|^{1/5} + C_2 A^2 \alpha^{4/5} |\mathbf{e} \cdot \mathbf{v}_{ij}|^{2/5} \mp \dots \quad (2.2)$$

with

$$\alpha = \left(\frac{3}{2}\right)^{3/2} \frac{Y \sqrt{R^{\text{eff}}}}{m^{\text{eff}} (1 - \nu^2)}, \quad (2.3)$$

with the effective mass and radius

$$m^{\text{eff}} = \frac{m_1 m_2}{m_1 + m_2}, \quad R^{\text{eff}} = \frac{R_1 R_2}{R_1 + R_2},$$

Young's modulus Y , Poisson's ratio ν , and a parameter A which depends on the dissipative parameters of the particle material (for details see Brilliantov *et al.* 1996). The numerical constants C_i are known analytically (Schwager & Pöschel 1998; Ramírez *et al.* 1999); the first two read $C_1 \approx 1.1534$ and $C_2 \approx 0.7982$.

Equation (2.2) refers to the case of pure viscoelastic interaction with linear stress-strain and stress-strain-rate relations. Deformation and deformation rate must be small to ensure the validity of the linear dependencies. Hence, the relative velocity $g = |\mathbf{v}_{ij} \cdot \mathbf{e}|$ should not be too large to fulfil these requirements. Obviously, the condition of linearity excludes plastic deformation of the material. Equation (2.2) has been derived in quasi-static approximation, which ignores the vibration and sound modes resulting from the collision (Brilliantov *et al.* 1996). This approximation also requires limitation for the relative velocity g from above, which is discussed in detail in Brilliantov *et al.* (1996). Finally, application of purely viscoelastic model implies that g is not too small so that the surface effects such as roughness, adhesion and van der Waals interactions may be neglected. The dependence of the form (2.2) for $\varepsilon = \varepsilon(g)$ was suggested by Kuwabara & Kono (1987), who used heuristic arguments.

3. Boltzmann equation for homogeneous granular gas

Consider a gas of identical granular particles of unit mass during the homogeneously cooling state. For simplicity, we will address rarefied granular gases, when one can ignore the increasing collision frequency due to the excluded volume effects (generalization at this point is straightforward) and non-locality of collisions in the collision

integral. For this case the time evolution of the velocity distribution function is described by the Boltzmann equation, which reads for the force-free case (Esipov & Pöschel 1997)

$$\frac{\partial}{\partial t} f(\mathbf{v}_1, t) = \sigma^2 \int d\mathbf{v}_2 \int d\mathbf{e} \Theta(-\mathbf{v}_{12} \cdot \mathbf{e}) |\mathbf{v}_{12} \cdot \mathbf{e}| \times \{\chi f(\mathbf{v}_1^{**}, t) f(\mathbf{v}_2^{**}, t) - f(\mathbf{v}_1, t) f(\mathbf{v}_2, t)\} \equiv I(f, f), \quad (3.1)$$

where $\sigma = 2R$ is the diameter of particles, $\Theta(x)$ is the Heaviside step-function. The velocities \mathbf{v}_1^{**} and \mathbf{v}_2^{**} refer to the pre-collisional velocities of the so-called inverse collision, which results with \mathbf{v}_1 and \mathbf{v}_2 as the after-collisional velocities. Finally, the factor χ in the gain term appears, respectively, from the Jacobian of the transformation $d\mathbf{v}_1^{**} d\mathbf{v}_2^{**} \rightarrow d\mathbf{v}_1 d\mathbf{v}_2$ and from the relation between the lengths of the collisional cylinders $\varepsilon |\mathbf{v}_{12}^{**} \cdot \mathbf{e}| dt = |\mathbf{v}_{12} \cdot \mathbf{e}| dt$ (Brilliantov & Pöschel 2000a).

(a) *Simplified collision model: $\varepsilon = \text{const.}$*

With $\varepsilon = \text{const.}$ one obtains $\chi = \varepsilon^{-2}$ (Goldshtein & Shapiro 1995; van Noije & Ernst 1998), i.e. χ is independent on the impact velocity and, therefore, independent on time. With the scaling ansatz for the distribution function (Esipov & Pöschel 1997),

$$f(\mathbf{v}, t) = \frac{n}{v_0^3(t)} \tilde{f}\left(\frac{v}{v_0(t)}\right) = \frac{n}{v_0^3(t)} \tilde{f}(\mathbf{c}), \quad (3.2)$$

where n is the number density of the granular gas and $v_0(t)$ is the thermal velocity, related to the granular temperature $T(t)$ as

$$\frac{3}{2} n T(t) = \int d\mathbf{v} \frac{1}{2} m v^2 f(\mathbf{v}, t) = \frac{3}{2} n \frac{1}{2} m v_0^2(t), \quad (3.3)$$

the Boltzmann equation decouples into two independent equations, one for the (time-independent) scaling function $\tilde{f}(\mathbf{c})$, and one for the time-dependent thermal velocity, i.e. for temperature (Esipov & Pöschel 1997). Solving the equation for the temperature one obtains $T(t) = T_0 / (1 + t/t_0)^2$.

The solution of the other equation for $\tilde{f}(\mathbf{c})$ may be found in terms of the Sonine polynomial expansion (Goldshtein & Shapiro 1995; van Noije & Ernst 1998; Brilliantov & Pöschel 2000b). For the case of $\varepsilon = \text{const.}$ it reads

$$\tilde{f}(\mathbf{c}) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p S_p(c^2) \right\}, \quad (3.4)$$

where $\phi(c) \equiv \pi^{-3/2} \exp(-c^2)$ is the Maxwellian distribution for the rescaled velocity. The first few Sonine polynomials are

$$S_0(x) = 1, \quad S_1(x) = -x^2 + \frac{3}{2}, \quad S_2(x) = \frac{1}{2}x^2 - \frac{5}{2}x + \frac{15}{8}. \quad (3.5)$$

The leading (zero-order) term in equation (3.4) is the Maxwellian distribution, while the next-order terms characterize the deviation of the distribution from the Maxwellian. Physically, the coefficients a_i measure the deviation of the moments of the

velocity distribution function $\langle c^i \rangle$ from those of the Maxwellian distribution, $\langle c^i \rangle_0$, e.g.

$$a_1 = (\langle c^2 \rangle - \langle c^2 \rangle_0) / \langle c^2 \rangle_0, \quad a_2 = (\langle c^4 \rangle - \langle c^4 \rangle_0) / \langle c^4 \rangle_0. \quad (3.6)$$

Thus, the Sonine polynomials expansion characterizes the form of the velocity distribution function in terms of its moments. For the case $\varepsilon = \text{const.}$, $\tilde{f}(\mathbf{c})$ is time independent, therefore, the coefficients a_i are time-independent constants too. Due to the definition of temperature, $a_1 = 0$ (Goldshtein & Shapiro 1995; van Noije & Ernst 1998; Brilliantov & Pöschel 2000b).

If inelasticity is small, one can restrict to the first non-vanishing term beyond the Maxwellian which is characterized by a_2 . For $\varepsilon = \text{const.}$ this coefficient reads (van Noije & Ernst 1998)

$$a_2 = \frac{16(1-\varepsilon)(1-2\varepsilon^2)}{9+24d+8\varepsilon d+41\varepsilon+30(1-\varepsilon)\varepsilon^2}. \quad (3.7)$$

A more accurate (although more complicated) expression for a_2 has been found by Brilliantov & Pöschel (2000b). This, however, coincides with (3.7) for small dissipation which is addressed here.

(b) *Realistic collision model: viscoelastic particles*

For viscoelastic particles the restitution coefficient ε depends on the impact velocity due to equation (2.2). Therefore, the factor χ in the Boltzmann equation (3.1) is not anymore constant as for $\varepsilon = \text{const.}$ but is time dependent too and, hence, the scaling ansatz (3.2) does not allow now to obtain a set of decoupled equations. Nevertheless, it is worth to use the generalized scaling ansatz,

$$f(\mathbf{v}, t) = \frac{n}{v_0^3(t)} \tilde{f}(\mathbf{c}, t), \quad (3.8)$$

which substituted into equation (3.1) yields after some algebra,

$$\frac{\mu_2}{3} \left(3 + c_1 \frac{\partial}{\partial c_1} \right) \tilde{f}(\mathbf{c}_1, t) + B^{-1} \frac{\partial}{\partial t} \tilde{f}(\mathbf{c}_1, t) = \tilde{I}(\tilde{f}, \tilde{f}), \quad (3.9)$$

where we define the dimensionless collisional integral,

$$\tilde{I}(\tilde{f}, \tilde{f}) = \int d\mathbf{c}_2 \int d\mathbf{e} \Theta(-\mathbf{c}_{12} \cdot \mathbf{e}) |\mathbf{c}_{12} \cdot \mathbf{e}| \{ \tilde{\chi} \tilde{f}(\mathbf{c}_1^{**}, t) \tilde{f}(\mathbf{c}_2^{**}, t) - \tilde{f}(\mathbf{c}_1, t) \tilde{f}(\mathbf{c}_2, t) \}, \quad (3.10)$$

with the reduced factor $\tilde{\chi}$,

$$\tilde{\chi} = 1 + \frac{11}{5} C_1 \delta' |\mathbf{c}_{12} \cdot \mathbf{e}|^{1/5} + \frac{66}{25} C_1^2 \delta'^2 |\mathbf{c}_{12} \cdot \mathbf{e}|^{2/5} + \dots, \quad (3.11)$$

which depends now on time via a quantity

$$\delta'(t) \equiv A \alpha^{2/5} [2T(t)]^{1/10} \equiv \delta [2T(t)/T_0]^{1/10}. \quad (3.12)$$

Here $\delta \equiv A \alpha^{2/5} [T_0]^{1/10}$ and T_0 is the initial temperature. We also define $B = B(t) \equiv v_0(t) \sigma^2 n$, and the moments of the dimensionless collision integral,

$$\mu_p \equiv - \int d\mathbf{c}_1 c_1^p \tilde{I}(\tilde{f}, \tilde{f}). \quad (3.13)$$

Since the velocity distribution function depends explicitly on time now, the coefficients of the Sonine expansion

$$\tilde{f}(\mathbf{c}, t) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p(t) S_p(c^2) \right\} \quad (3.14)$$

depend on time too. Inserting equation (3.14) into equation (3.9) one obtains equations to determine the Sonine coefficients $a_k(t)$. Technically, this is done by multiplying both sides of equation (3.9) with c_1^p and integrating over $d\mathbf{c}_1$ which yields an infinite set of equations for a_k , $k = 1, 2, 3, \dots$ (see Brilliantov & Pöschel (2000a) for details). With the assumption of small dissipation, the deviation of the distribution function from the Maxwellian may be well described by the first non-trivial Sonine coefficient a_2 only, i.e. by the fourth moment of the distribution (see equation (3.6)). The time evolution of this coefficient was derived in Brilliantov & Pöschel (2000a):

$$\frac{da_2}{dt} = \frac{4}{3} B \mu_2 (1 + a_2) - \frac{4}{15} B \mu_4. \quad (3.15)$$

Using the definitions of temperature, equation (3.3), of the second moment μ_2 , equation (3.13), and $B = v_0(t) \sigma^2 n$, which is proportional to the collision frequency (see equation (5.4)), the evolution of temperature reads

$$\frac{dT}{dt} = \frac{1}{3} \sigma^2 n v_0^3 \int d\mathbf{c}_1 c_1^2 \tilde{I}(\tilde{f}, \tilde{f}) = -\frac{2}{3} B T \mu_2. \quad (3.16)$$

With our approximation $\tilde{f} = \phi(c)[1 + a_2(t) S_2(c^2)]$ the time-dependent moments $\mu_p(t)$ defined by equation (3.13) may be expressed in terms of a_2 (van Noije & Ernst 1998; Brilliantov & Pöschel 2000b):

$$\begin{aligned} \mu_p = & -\frac{1}{2} \int d\mathbf{c}_1 \int d\mathbf{c}_2 \int d\mathbf{e} \Theta(-\mathbf{c}_{12} \cdot \mathbf{e}) |\mathbf{c}_{12} \cdot \mathbf{e}| \phi(c_1) \phi(c_2) \\ & \times \{1 + a_2[S_2(c_1^2) + S_2(c_2^2)] + a_2^2 S_2(c_1^2) S_2(c_2^2)\} \Delta(c_1^p + c_2^p), \end{aligned} \quad (3.17)$$

where $\Delta\psi(\mathbf{c}_i) \equiv [\psi(\mathbf{c}_i^*) - \psi(\mathbf{c}_i)]$ denotes the change of some function $\psi(\mathbf{c}_i)$ in the direct collision. Calculations up to second order in δ yield (Brilliantov & Pöschel 2000a)

$$\mu_2 = \sum_{k=0}^2 \sum_{n=0}^2 \mathcal{A}_{kn} \delta^{kn} a_2^n, \quad \mu_4 = \sum_{k=0}^2 \sum_{n=0}^2 \mathcal{B}_{kn} \delta^{kn} a_2^n, \quad (3.18)$$

where \mathcal{A}_{kn} and \mathcal{B}_{kn} are numerical coefficients which we write in a compact matrix form as (rows refer to the first index)

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ \omega_0 & \frac{6}{25} \omega_0 & \frac{21}{2500} \omega_0 \\ -\omega_1 & -\frac{119}{400} \omega_1 & -\frac{4641}{640000} \omega_1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 4\sqrt{2\pi} & \frac{1}{8}\sqrt{2\pi} \\ \frac{56}{10} \omega_0 & \frac{1806}{250} \omega_0 & \frac{567}{12500} \omega_0 \\ -\frac{77}{10} \omega_1 & -\frac{149054}{13750} \omega_1 & -\frac{348424}{5500000} \omega_1 \end{pmatrix},$$

$$\omega_0 \equiv 2\sqrt{2\pi} 2^{1/10} \Gamma\left(\frac{21}{10}\right) \approx 6.48562, \quad \omega_1 \equiv \sqrt{2\pi} 2^{1/5} \Gamma\left(\frac{16}{5}\right) C_1^2 \approx 9.28569.$$

Thus, equations (3.15) and (3.16), together with equations (3.18) and (3.12), constitute a closed set of equations to find the time evolution of temperature and of a_2 .

In contrast to the case $\varepsilon = \text{const.}$, which implies $a_2 = \text{const.}$, in a gas of viscoelastic particles the time evolution of temperature is coupled to the time evolution of the Sonine coefficient a_2 . Physically, this means that not only temperature but also the shape of the velocity distribution function changes with time. Therefore, the state of a granular gas of realistic (viscoelastic) particles cannot be described by temperature alone but further time-dependent quantities are needed which characterize the form of the distribution function.

(c) *Small dissipation limit: adiabatic approximation for $a_2(t)$*

The coupled equations for temperature (3.16) and the second Sonine coefficient (3.15) may be solved analytically in the limit of small dissipation δ yielding (see Brilliantov & Pöschel (2000a) for details)

$$\frac{T(t)}{T_0} = \left(1 + \frac{t}{\tau_0}\right)^{-5/3}, \quad (3.19)$$

where we introduce the characteristic time,

$$\tau_0^{-1} = \frac{16}{5}q_0\delta\tau_c(0)^{-1} = \frac{16}{5}q_0\delta 4\sigma^2 n\sqrt{\pi T_0/m}, \quad (3.20)$$

with $\tau_c(0)$ being the initial mean-collision time and we define a constant

$$q_0 = 2^{1/5}\Gamma\left(\frac{21}{10}\right)C_1/8 \approx 0.173\,318.$$

The coefficient a_2 evolves for small δ as

$$a_2(t) = -\frac{12}{5}w(t)^{-1}\{\text{Li}[w(t)] - \text{Li}[w(0)]\}, \quad (3.21)$$

where $w(t) \equiv \exp[(q_0\delta)^{-1}(1 + t/\tau_0)^{1/6}]$, and $\text{Li}(x) = \int_0^x [\ln(t)]^{-1} dt$ is the logarithmic integral (Brilliantov & Pöschel 2000a).

The analysis of the obtained time dependence for the second Sonine coefficient a_2 shows that his time evolution exhibits two different stages: (1) fast initial relaxation on the scale of the mean collision time *ca.* $\tau_c(0)$, and (2) subsequent slow evolution on the time-scale *ca.* $\tau_0 \gg \tau_c(0)$, which refers to the time-scale of the temperature evolution (figure 1). Asymptotically, $a_2(t)$ approaches zero due to a power law (Brilliantov & Pöschel 2000a). Due to the coupling between temperature and a_2 , the form of the velocity distribution function changes in accordance with temperature.

For the hydrodynamic description of the gas, we can restrict the analysis to the second stage of the evolution. In adiabatic approximation we neglect the term da_2/dt in (3.15) and assume that a_2 is determined by the instant values of μ_2 and μ_4 . In this approximation the differential equation (3.15) reduces to an algebraic equation for a_2 , since μ_2 and μ_4 depend only on a_2 and current temperature (see equations (3.18) and (3.12)). In other words, a_2 (and thus the shape of the velocity distribution function) is assumed to follow adiabatically temperature, so that it immediately adjusts the altering temperature. Within the adiabatic approximation one obtains for the coefficient a_2 (Brilliantov & Pöschel 2001, unpublished work)

$$a_2 = a_{21}\delta(2T/T_0)^{1/10} + a_{22}\delta^2(2T/T_0)^{1/5} + \dots, \quad (3.22)$$

where a_{21} , a_{22} , are pure numbers: $a_{21} = -3\omega_0/20\sqrt{2\pi} \approx -0.388\,11$, $a_{22} \approx -11.509$.

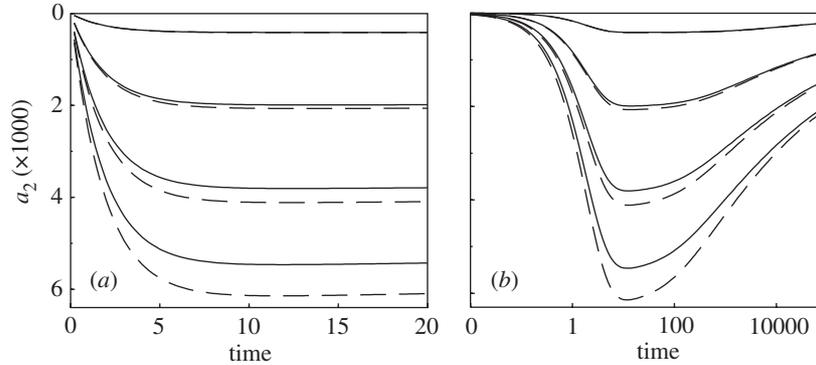


Figure 1. Time dependence of the second coefficient of the Sonine polynomial expansion $a_2(t) \times 1000$ (solid and dashed lines are, respectively, numerical and analytical solutions). Time is given in units of the mean collision time $\tau_c(0)$. $\delta = 0.001, 0.005, 0.01, 0.015$ (top to bottom). (a) First stage of the evolution, i.e. quick relaxation within few mean collision times. (b) Second stage, i.e. saturation and subsequent slow relaxation back to zero due to the coupling between temperature and $a_2(t)$.

4. Hydrodynamic equations and kinetic coefficients

For the case of realistic (viscoelastic) particles which interact via a velocity-dependent restitution coefficient, the time evolution of temperature is generally coupled to the time evolution of the Sonine coefficients, as discussed in detail in the previous section. The Sonine coefficients characterize the even moments of the velocity distribution function, i.e. its time-dependent shape. Therefore, in general, the hydrodynamic description of the system requires an extended set of fields: the conventional set of fields consist of density, $n(\mathbf{r}, t)$, velocity, $\mathbf{u}(\mathbf{r}, t)$, and temperature, $T(\mathbf{r}, t)$, which refer correspondingly to zeroth-, first- and second-order moments of the distribution function,

$$n(\mathbf{r}, t) = \int d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) = \langle v^0 \rangle, \quad (4.1)$$

$$n(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t) = \int d\mathbf{v} \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) = \langle \mathbf{v} \rangle, \quad (4.2)$$

$$\frac{3}{2}n(\mathbf{r}, t)T(\mathbf{r}, t) = \int d\mathbf{v} \frac{1}{2}mV^2 f(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2}m\langle V^2 \rangle, \quad (4.3)$$

with $\mathbf{V} \equiv \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$. For the description of a granular gas one needs to consider next-order moments, such as $\langle V^4 \rangle$, $\langle V^6 \rangle$, etc., which describe the shape of the distribution function, since these high-order moments also depend on time and enter the evolution equation for temperature. In principle, this may be done within the Grad approach (see, for example, Ramírez *et al.* 2000; Grad 1960); however, the resulting equations are rather cumbersome.

As before, we assume that dissipation is small, so that one can neglect high-order Sonine coefficients a_k for $k > 2$ and use the adiabatic approximation for a_2 (see § 3c). In this approximation the form of the distribution function is characterized by a_2 only and is, therefore, completely described by the current temperature. Hence, the above set of fields, n , \mathbf{u} , T is closed, since the equations for these fields do not contain independent higher-order moments.

Multiplying the Boltzmann equation for inhomogeneous gas,

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla\right) f(\mathbf{r}, \mathbf{v}_1, t) = I(f, f), \quad (4.4)$$

correspondingly with v_1^0 , \mathbf{v}_1 , v_1^2 , and integrating over $d\mathbf{v}_1$, one obtains the hydrodynamic equations for n , \mathbf{u} and T (see, for example, Chapman & Cowling 1970),

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \quad (4.5)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + (nm)^{-1} \nabla \cdot \hat{P} = 0, \quad (4.6)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T + \frac{2}{3n} (\hat{P} : \nabla \mathbf{u} + \nabla \cdot \mathbf{q}) + \zeta T = 0, \quad (4.7)$$

with the coefficient ζ in the sink term ζT ,

$$\begin{aligned} & -\zeta(\mathbf{r}, t) \\ &= \frac{\sigma^2 m}{12nT} \int d\mathbf{v}_1 d\mathbf{v}_2 \int d\mathbf{e} \Theta(-\mathbf{v}_{12} \cdot \mathbf{e}) |\mathbf{v}_{12} \cdot \mathbf{e}| f(\mathbf{r}, \mathbf{v}_1, t) f(\mathbf{r}, \mathbf{v}_2, t) (\mathbf{v}_{12} \cdot \mathbf{e})^2 (1 - \varepsilon^2), \end{aligned}$$

and with the pressure tensor \hat{P} and the heat flux \mathbf{q} expressed in terms of the fields gradients:

$$P_{ij} = p\delta_{ij} - \eta(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij} \nabla \cdot \mathbf{u}), \quad (4.8)$$

$$\mathbf{q} = -\kappa \nabla T - \mu \nabla n. \quad (4.9)$$

Here $p = nT$ is the isotropic (hydrostatic pressure); η , κ and μ are phenomenological coefficients, for which microscopic expressions may be obtained. While η and κ have the conventional meaning of viscosity and thermal conductivity as for the molecular gases, the new coefficient μ does not have any corresponding analogue (Brey *et al.* 1998; Brey & Cubero 2000). Physically, μ describes the additional heat flux between dense and rarefied regions of the gas, which arises because the former cool down more quickly than the latter ones owing to the increasing collision frequency (Goldhirsch & Zanetti 1993).

The form of the hydrodynamic equations (up to additional terms ζT and $\mu \nabla n$) is the same as for molecular gases, and the particular dependence on inelasticity is hidden in the kinetic coefficients and in the cooling rate ζ . Calculation of these coefficients may be performed within the Chapman–Enskog approach (Chapman & Cowling 1970), where the velocity distribution function is found as an expansion,

$$f = f^{(0)} + \lambda f^{(1)} + \lambda^2 f^{(2)} + \dots, \quad (4.10)$$

with each power of λ corresponding to the order of the spatial gradient. Thus, $f^{(0)}$ denotes the zero-gradient solution (which refers to the homogeneous cooling state in the context of granular gas), $f^{(1)}$ corresponds to the solution obtained in the linear-order approximation with respect to the fields gradients, $f^{(2)}$ to the second-order approximation, etc. Similarly, as for molecular gases, it is assumed that the space and time dependence of the velocity distribution function appears only through the time and space dependence of the hydrodynamic fields.

As mentioned before for the case of the constant restitution coefficient $\varepsilon = \text{const.}$, the shape of $\tilde{f}(\mathbf{c})$ depends only on ε . Contrarily, for a gas of viscoelastic particles the restitution coefficient depends on impact velocity, i.e. on temperature and, therefore, on time. Hence, the shape of the velocity distribution function $\tilde{f}(\mathbf{c}, t)$ depends on time too. In an adiabatic approximation a_2 depends solely on the current temperature, so that one can write for the granular gas of viscoelastic particles (Brilliantov & Pöschel 2001, unpublished work)

$$f(\mathbf{r}, \mathbf{v}, t) = f[\mathbf{v}; n(\mathbf{r}, t), \mathbf{u}(\mathbf{r}, t), T(\mathbf{r}, t), a_2(T(\mathbf{r}, t))]. \quad (4.11)$$

Note, that in the case of interest, the distribution has an additional space-time dependence through a_2 .

The kinetic coefficients and the cooling rate can be evaluated using the Chapman–Enskog approach with some necessary modifications concerning the time-dependence of the zero-order solution (which refers to the homogeneous cooling state) and due to an additional time-dependence of the collision integral, owing the impact velocity dependence of ε . The details of the calculation will be published elsewhere (N. V. Brilliantov & T. Pöschel 2001, unpublished work), here we present only the final result: the cooling rate can be found as an expansion in the time-dependent dissipative parameter $\delta' = \delta(2T/T_0)^{1/10}$,

$$\zeta^{(0)} = \sqrt{8T/9m\sigma^2} n[\omega_0 \delta' - \omega_2 \delta'^2 + \dots], \quad (4.12)$$

with the pure numbers $\omega_0 \approx 6.48562$ and $\omega_2 = \omega_1 + (9\sqrt{2\pi}/500\pi)\omega_0^2 \approx 9.88979$.

Similarly, we find for the kinetic coefficients,

$$\eta = \eta_0(1 + \delta' \tilde{\eta}_1 + \delta'^2 \tilde{\eta}_2 + \dots), \quad (4.13)$$

$$\kappa = \kappa_0(1 + \delta' \tilde{\kappa}_1 + \delta'^2 \tilde{\kappa}_2 + \dots), \quad (4.14)$$

$$\mu = \frac{\kappa_0 T}{n} (\delta' \tilde{\mu}_1 + \delta'^2 \tilde{\mu}_2 + \dots), \quad (4.15)$$

where

$$\eta_0 = \frac{5}{16\sigma^2} \left(\frac{mT}{\pi} \right)^{1/2} \quad \text{and} \quad \kappa_0 = \frac{75}{64\sigma^2} \left(\frac{T}{\pi m} \right)^{1/2} \quad (4.16)$$

are the kinetic coefficients for a gas of elastic particles, and the other coefficients are pure numbers, e.g.

$$\left. \begin{aligned} \tilde{\eta}_1 &= \frac{359}{3840} \sqrt{\frac{2}{\pi}} \omega_0 \approx 0.483791, \quad \tilde{\eta}_2 \approx 0.19295, \\ \tilde{\kappa}_1 &= \frac{7349}{19200} \sqrt{\frac{2}{\pi}} \omega_0 \approx 1.98071, \quad \tilde{\kappa}_2 \approx -24.2833, \\ \tilde{\mu}_1 &= \frac{19}{80} \sqrt{\frac{2}{\pi}} \omega_0 \approx 1.22901, \quad \tilde{\mu}_2 \approx -13.2615. \end{aligned} \right\} \quad (4.17)$$

The main difference between the obtained transport coefficients for the gas of viscoelastic particles and that with a constant restitution coefficient (see, for example, Brey *et al.* 1998; Brey & Cubero 2000) is their temperature dependence. In an expansion of these coefficients with respect to the small dissipative parameter δ , the temperature dependence of η and κ is the same in the leading term in both cases,

and scales as *ca.* \sqrt{T} . The cooling rate ζ and the transport coefficient μ , however, reveal a quite different dependence. Namely,

$$\begin{aligned} \text{for } \varepsilon = \text{const.}: \quad & \zeta \sim T^{1/2}, \quad \mu/T \sim T^{1/2}, \\ \text{for } \varepsilon = \varepsilon(g): \quad & \zeta \sim T^{3/5}, \quad \mu/T \sim T^{3/5}. \end{aligned}$$

Although the difference in the temperature dependence of μ is not very important, that for the cooling rate leads to a drastic change in the global behaviour of the granular gas, as discussed in the next section.

5. Structure formation in a granular gas of viscoelastic particles

The set of nonlinear partial differential equations which describes the hydrodynamics of a granular gas is certainly too complicated to be analysed analytically. Therefore, we have to apply some simplifications. First we linearize the equations with respect to the homogeneously cooling state,

$$T(\mathbf{r}, t) = T_h(t) + \delta T(\mathbf{r}, t), \quad (5.1)$$

$$n(\mathbf{r}, t) = n_h + \delta n(\mathbf{r}, t), \quad (5.2)$$

with a constant number density $n_h = N/V$ and space-uniform, but time-dependent temperature $T_h(t)$, and assume that deviations of the hydrodynamic fields from their values in the homogeneously cooling state is small, i.e. we assume that

$$\theta \equiv \frac{\delta T}{T_h} \ll 1, \quad \rho \equiv \frac{\delta n}{n_h} \ll 1, \quad |\mathbf{w}| \equiv \frac{|\mathbf{u}|}{v_0} \ll 1. \quad (5.3)$$

For the (small) quantities $\theta(\mathbf{r}, t)$, $\rho(\mathbf{r}, t)$, $\mathbf{w}(\mathbf{r}, t)$ one can derive linear partial differential equations. For further simplification, we perform a Fourier transform, which converts the partial differential equations into ordinary ones for the Fourier amplitudes (modes) $\theta_{\mathbf{k}}(t)$, $\rho_{\mathbf{k}}(t)$, $\mathbf{w}_{\mathbf{k}}(t)$. If we substitute time t by a dimensionless time, measured in the accumulated number of collisions per particle,

$$d\tau = \frac{8}{5} \frac{dt}{\tau_c(t)}, \quad \tau_c(t)^{-1} = 2\sqrt{2\pi}n\sigma^2v_0(t), \quad (5.4)$$

and distance \mathbf{r} by its dimensionless analogue, measured in units of the mean-free path,

$$d\hat{\mathbf{r}} = \frac{8}{5} \frac{d\mathbf{r}}{l_0}, \quad l_0^{-1} = 2\sqrt{2\pi}n\sigma^2, \quad (5.5)$$

we arrive at a simplified set of linearized equations,

$$\frac{\partial \mathbf{w}_{\mathbf{k}\perp}}{\partial \tau} = [b\delta'(\tau) - k^2]\mathbf{w}_{\mathbf{k}\perp}, \quad (5.6)$$

$$\frac{\partial \rho_{\mathbf{k}}}{\partial \tau} = -ikw_{\mathbf{k}\parallel}, \quad (5.7)$$

$$\frac{\partial \theta_{\mathbf{k}}}{\partial \tau} = -[\frac{5}{2}\tilde{\mu}_1k^2 + 2b]\delta'(\tau)\rho_{\mathbf{k}} - [\frac{5}{2}k^2 + \frac{6}{5}b\delta'(\tau)]\theta_{\mathbf{k}} - \frac{2}{3}ikw_{\mathbf{k}\parallel}, \quad (5.8)$$

$$\frac{\partial w_{\mathbf{k}\parallel}}{\partial \tau} = -\frac{1}{2}ik\rho_{\mathbf{k}} - \frac{1}{2}ik\theta_{\mathbf{k}} + [b\delta'(\tau) - \frac{4}{3}k^2]w_{\mathbf{k}\parallel}, \quad (5.9)$$

with a pure number $b = 5\omega_0/(48\sqrt{2\pi})$ and with a time-dependent dissipation,

$$\delta'(\tau) = 2^{1/10}\delta(1 + q_0\delta\tau/3)^{-1}. \quad (5.10)$$

If one compares the set (5.6)–(5.9) of linearized equations for Fourier modes derived for a gas of viscoelastic particles with the analogue set of simplified particles, interacting with $\varepsilon = \text{const.}$, one notices an important difference. For the case $\varepsilon = \text{const.}$, all the coefficients, η , κ , μ/T , and the cooling rate ζ , have identical temperature dependence, *ca.* $T^{1/2}$. Therefore, in this case the dependence of the transport coefficients on the (rescaled) time drops out, while for viscoelastic interaction at least some of the coefficients depend explicitly on the rescaled time. This is due to the fact that for viscoelastic particles μ/T and ζ do not scale as $T^{1/2}$ but depend differently on T . For this reason it is not possible to eliminate the time dependence of the coefficients by time and length transformations.

Obviously, within the linear theory, equation (5.6) describing the transverse mode $\mathbf{w}_{\mathbf{k}\perp}$, which corresponds to the component of the flow field perpendicular to the wave-vector \mathbf{k} , decouples from the other equations. Its solution reads

$$\mathbf{w}_{\mathbf{k}\perp}(\tau) = \mathbf{w}_{\mathbf{k}\perp}(0)(1 + q_0\delta\tau/3)^5 e^{-k^2\tau}, \quad (5.11)$$

where $\mathbf{w}_{\mathbf{k}\perp}(0)$ is the initial amplitude of the shear mode, which appears due to fluctuations. There is a critical value,

$$k_{\perp}^* \equiv \sqrt{5q_0\delta/3}, \quad (5.12)$$

which separates two different scenarios: (1) for $k \geq k_{\perp}^*$ the mode decays, and (2) for $k < k_{\perp}^*$ it grows until it reaches a maximum,

$$w_{\mathbf{k}\perp}^{max} = w_{\mathbf{k}\perp}(0) \left(\frac{5q_0\delta}{3k^2e} \right)^5 \exp\left(\frac{3k^2}{q_0\delta}\right) \quad \text{at time } \tau_{k\perp}^* = \frac{5}{k^2} - \frac{3}{q_0\delta}. \quad (5.13)$$

After this moment it decreases and completely dies off. The growth of the shear modes corresponds to the vortex formation. According to our linear analysis, the vortices of size *ca.* k^{-1} would decay after the transient time *ca.* $\tau_{k\perp}^*$. Since any finite-sized system, of the characteristic size L , has the minimal wavenumber *ca.* $2\pi/L$, all shear modes will decay by the time *ca.* $5L^2/4\pi^2 - 3/q_0\delta$. That means that the vortex formation for any finite-size system is just a transient process, so that the vortices tend to dissolve as the granular gas evolves further.

This conclusion (based on the linear analysis) is in sharp contrast to the predictions for a granular gas with $\varepsilon = \text{const.}$: once created, the vortices remain persistent throughout the evolution of the granular gas.

The same analysis may be applied to the other three modes. To this end one writes equations (5.7)–(5.9) in matrix form,

$$\left(\frac{\partial \rho_{\mathbf{k}}}{\partial \tau}, \frac{\partial \theta_{\mathbf{k}}}{\partial \tau}, \frac{\partial w_{\mathbf{k}\parallel}}{\partial \tau} \right)^T = \hat{\mathbf{M}}_{\mathbf{k}}(\tau) (\rho_{\mathbf{k}}, \theta_{\mathbf{k}}, w_{\mathbf{k}\parallel})^T, \quad (5.14)$$

where the superscript ‘T’ denotes the transpose. The eigenvalues ξ_i of this matrix depend on τ and may be found by the numerical solution of the cubic equation, $\det |\hat{\mathbf{M}}_{\mathbf{k}}(\tau) - \hat{I}\xi| = 0$ (see, for example, van Noije & Ernst 2000).

Just as in the case $\varepsilon = \text{const.}$, this yields three eigenmodes: the heat mode and two other modes, which in the elastic case correspond to sound modes. The real parts of the eigenvalues of the sound modes are always negative. For the heat mode one obtains the time-dependent frontier wavenumber

$$k_{\parallel}^*(\tau) \simeq \frac{2^{31/20}b}{5} \left(\frac{\delta}{1 + \delta q_0 \tau/3} \right)^{1/2}, \quad (5.15)$$

such that for $k > k_{\parallel}^*(0)$ the heat mode always decays, while for $k < k_{\parallel}^*(0)$ it initially grows. Since $k_{\parallel}^*(\tau)$ decreases with time, for any finite-size system, it becomes smaller than the minimal wavenumber $2\pi/L$. This occurs after a time

$$\tau_{\parallel}^* \approx (2^{1/10}6b/25q_0\pi^2)L^2.$$

For $\tau > \tau_{\parallel}^*$ the amplitude of any heat mode in the system starts to decay. Again we see (within the linear analysis) that the same scenario as for the shear mode takes place. Any initial fluctuation with $k < k_{\parallel}^*(0)$, which refers to the heat mode (i.e. to some combination of $\theta_{\mathbf{k}}(t)$, $\rho_{\mathbf{k}}(t)$ and $w_{\mathbf{k}\parallel}(t)$) first increases, and then dies off. The interval of time when the heat mode grows corresponds to the enhancement of the density inhomogeneity, i.e. to clustering. Thus, we conclude that clusters with sizes larger than *ca.* $1/k_{\parallel}^*(0)$ start initially to emerge in the system, but then, at a later time, they dissolve.

This suggests (at least on the level of the linear analysis) that pattern formation such as clustering or vortex formation, which has been observed for the simplified model of $\varepsilon = \text{const.}$, occurs in granular gases of particles with realistic collisional model only as transient states.

6. Conclusion

We derived the hydrodynamic equations for a granular gas of viscoelastic particles and analysed their stability in the limits of a linear approximation. In contrast to the common assumption of a constant coefficient of restitution $\varepsilon = \text{const.}$, which contradicts experiments and even a dimension analysis, realistic collisions (as for viscoelastic particles) imply a coefficient which depends on the impact velocity, $\varepsilon = \varepsilon(g)$.

To derive the hydrodynamic equations we start from the Boltzmann equation for the homogeneous cooling state and find the velocity distribution function. Unlike for $\varepsilon = \text{const.}$, for viscoelastic particles the velocity distribution function reveals a time dependence of its shape, which for small dissipation may be characterized by the second Sonine coefficient a_2 . The time evolution of a_2 is coupled to the time evolution of temperature.

Generally, this requires an extension of the common set of hydrodynamic fields, namely density, temperature and flow-field. For small-inelasticity we perform the adiabatic approximation for a_2 , i.e. we assume that the shape of the velocity distribution function immediately adjusts to the current temperature. The adiabatic approximation allows us to remain within the unextended set of hydrodynamic fields. Using the Chapman–Enskog scheme we derive the hydrodynamic equations and find microscopic expressions for the transport coefficients.

The stability analysis of the linearized hydrodynamic equations reveals that all shear and heat modes ultimately die out after some transient processes. Since vortex

formation and clustering in granular gases is attributed to the shear and heat mode instability, we conclude that in a gas of viscoelastic particles those structures may develop only during transient states; however, asymptotically the system tends to the flow-free stage with homogeneous density. Although we cannot judge whether the omitted nonlinear terms stabilize clusters and vortexes, physical arguments support the suggested scenario: indeed, since the collisions become more and more elastic with time, the system tends to behave as an assembly of elastic particles, i.e. it tends to become homogeneous and flow-free. During the first stage of its evolution, however, the gas is characterized by structure formation—the salient feature of dissipative systems. This behaviour is in contrast to the case of a simplified granular gas with $\varepsilon = \text{const.}$: here the shear and heat modes infinitely grow for all wavenumbers below a certain threshold values.

Our main conclusion is based on a linear analysis and on the assumption of small inelasticity. By now it is not clear what happens for large inelasticity and what would be the impact of the omitted nonlinear terms.

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