Journal of Statistical Physics, Vol. 84, Nos. 5/6, 1996

Avalanche Statistics of Sand Heaps

Volkhard Buchholtz[1] and Thorsten Pöschel[1]

Received January 6, 1996

Large-scale computer simulations are presented to investigate the avalanche statistics of sandpiles using molecular dynamics. We show that different methods of measurement lead to contradictory conclusions, presumably due to avalanches not reaching the end of the experimental table.

KEY WORDS: Granular material; self-organized criticality.

The physics of an evolving sandpile has been of great interest to physicists and engineers and much work has been done in this field. One of the most popular (or sometimes unpopular) ideas is the concept of self-organized criticality (SOC).[1] It has been argued by many physicists that sandpiles can be described by cellular automata in two or three dimensions (e.g., ref. 2) and by stochastic cellular automata (e.g., ref. 3) which in simulations might show SOC behavior. There are many effects in nature which are supposed to reveal SOC, and hence a variety of articles have investigated its theory (e.g., ref. 4). When particles are dropped one after the other onto the top of a sand heap one observes avalanches. The time intervals between successive avalanches and the size distribution of the avalanches have been of interest to experimentalists as well as theorists, and there is a controversy over whether they obey a power law or not.[5,6]

In an experimental work Jaeger *et al.*[7] investigated the avalanche sizes of a pile contained in a box with one open side. The material flow over the edge of the box was measured between a pair of capacitor plates. From the fluctuation of the capacity they concluded that the sizes of the avalanches might *not* be power-law distributed. To determine the size of an

[1] Humboldt-Universität zu Berlin, Institut für Physik, Invaliderstraße 110, D-10115 Berlin, Germany; e-mail: volkhard@itp02.physik.hu-berlin.de, thorsten@itp02.physik.hu-berlin.de; URL: http://summa.physik.hu-berlin.de/.

822/84/5-6-32

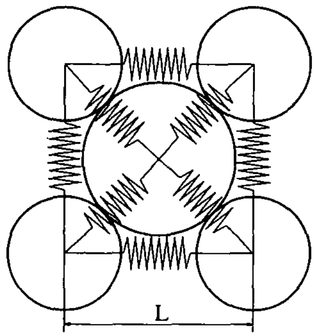

Fig. 1. Each of the nonspherical particles consists of five spheres.

avalanche they measure the capacity change, i.e., the mass of the particles which fall over the edge of the table. Bretz *et al.*[8] measured the avalanche distribution by recording the temporal behavior of the inclination of the heap's surface. In contrast to ref. 7, results support the hypothesis of the power-law distribution.

We want to present the results of a large-scale computer experiment where we recorded the distribution of the sizes of avalanches using both methods. We will show that possibly the two measurements[7,8] do not contradict, but support each other.

In a previous paper[9] we showed that in a simulation using two-dimensional molecular dynamics of nonspherical particles one can find a power-law behavior of the size distribution of the avalanches. Our particles k are built up of five spheres which are connected by springs (Fig. 1).

At rest the inner sphere touches the surrounding spheres of the same particle. For details of the forces acting between the spheres of the same grain via springs and the forces acting between colliding grains see ref. 10.

Using molecular dynamics, we build up the pile by dropping the particles one after the other on the top of the evolving pile. A particle is dropped when all avalanches and fluctuations caused by the previously released grain have faded away, i.e., we wait until the maximum velocity of the particles comes very close to zero. Then the inclination of the heap is measured by the following procedure: Suppose the shape of the heap of height H built up on a surface of width B is close to a triangle. Then its slope is given by

$$\Phi = \arctan\left[\frac{H - (1/M)\sum_{k=1}^{N} m^{(k)} y^{(k)}}{(2/M)\sum_{k=1}^{N} m^{(k)} x^{(k)}}\right] \qquad (1)$$

where $m^{(k)}$ and $x^{(k)}$ and $y^{(k)}$ are the mass and the position, respectively, of the kth grain and M is the sum of the masses of all particles $M = \sum_{k=1}^{N} m^{(k)}$. Since our heap is close to, but not an ideal triangle we calculate the height H using

$$H = \frac{2}{x_{\max}} \int_0^{x_{\max}} h(x)\, dx \tag{2}$$

where $x_{\max}$ is the x position of the grain which is closest to the end of the table. From the fluctuations of the slope according to Eq. (1) we can conclude the approximate size of the avalanche according to a decrease in the slope:

$$\Delta^{(1)}M = \frac{B^2}{2}\,\rho(\tan\phi' - \tan\Phi) \tag{3}$$

where Φ' is the slope before the dropping event, $B \approx P = 30.7$ cm is the length of the table, and $\rho = 0.59 g \cdot \mathrm{cm}^{-2}$ denotes the average density of the heap. This method for the measurement of the size of an avalanche (indexed by $\Delta^{(1)}M$) is close to the experimental method used by Bretz *et al.*[8] Another method, which was used by Jaeger *et al.*[7] and by Rosendahl *et al.*[6] is to measure the weight of the material which reaches the end of the finite table during an avalanche using a balance or a capacitor. They considered the weight of material flowing over the border of the table to be the size of the avalanche $\Delta^{(2)}M$. As in the experiment in the molecular dynamics simulations, we calculated the mass of the particles which reach the end of the table, i.e., $x^{(k)} > P$.

Collecting the idle time of all the computers of our department over a period of about 1 year, we found enough computer power to perform a large-scale molecular dynamics simulation. The heap was built up on a rough surface of width $P = 30.7$ cm $\approx 96 \langle L^{(k)} \rangle$. The radii $r_i^{(k)}$ of the outer spheres of the particles k were equally distributed in the interval $r_i^{(k)} \in (0.05, 0.11)$ cm, while the radii of the inner spheres are determined by the relation $r_m^{(k)} = L^{(k)}/\sqrt{2} - r_i^{(k)}$ $(i = 1,..., 4)$ where $L^{(k)}$ is the size of the kth grain (Fig. 1). In ref. 10, Fig. 10, we show that the relation $L^{(k)}/r_i^{(k)} = 4$ reproduces well the static friction behavior of sandpiles. The average number of particles on the heap was $N_{\mathrm{av}} = 930$.

Figure 2 shows the time series of the avalanche size from both procedures, $\Delta^{(1)}M$ and $\Delta^{(2)}M$. The size distributions of the data shown in Fig. 2 are drawn in Fig. 3. We find that the distribution of the avalanche sizes measured by means of the fluctuations in the slope $\Delta^{(1)}M$ reveals a typical power-law behavior for avalanches smaller than $\Delta^{(1)}M < 2$ g, while the distribution according to the direct measurement of the avalanche sizes $\Delta^{(2)}M$ does *not* show a power-law behavior. We claim that the difference illustrated in Fig. 3 comes from the fact that the second method is not able

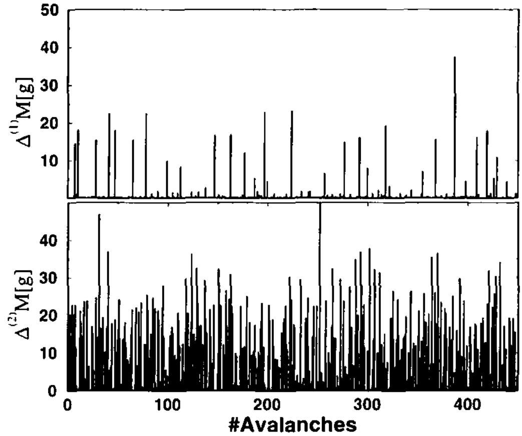

Fig. 2. Series of 450 avalanches. Top: the avalanche size $\Delta^{(1)}M$ concluded from the fluctuation in slope [Eq. (3)]; bottom: the avalanche size $\Delta^{(2)}M$ calculated from the mass of particles that reach the end of the table. The fraction of small avalanches is much higher for the upper figure.

to account for those avalanches which do not reach the end of the table. Obviously the larger the pile, the higher is the fraction of avalanches which do not reach the border of the table. This coincides with the observations by Jaeger *et al.*,[7] who found a deviation from the power law scaling for *large* systems. They found a sharply peaked avalanche distribution of large, system-overspanning avalanches.

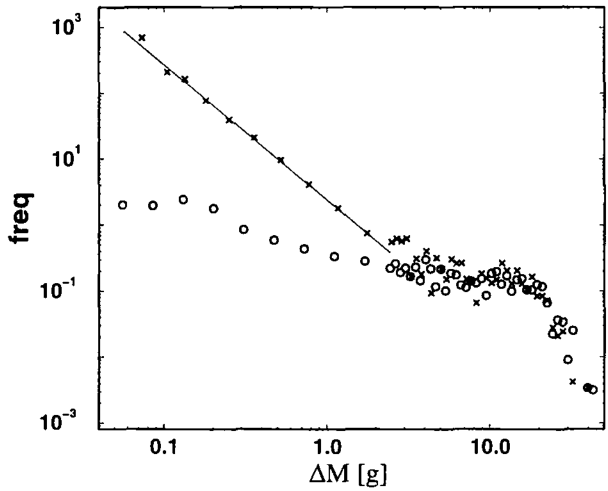

Fig. 3. The size distribution of the avalanches on a log–log scale for $\Delta^{(1)}M$ (×) and $\Delta^{(2)}M$ (○). The sizes are measured in grams. The line shows the function freq $\sim \Delta^{(1)}M^{-1.85}$.

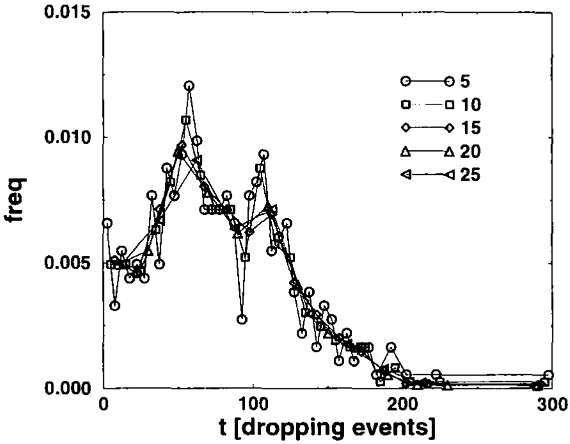

Fig. 4. The waiting-time distribution of the intervals between two consecutive large avalanches. The figure shows the number of pairs of avalanches versus their time interval measured in dropping events. The double-peak structure is preserved for different discretization interval sizes.

For large avalanches in our simulation both types of measurements lead to very similar results, which supports our conclusion. In agreement with the experimental observations by Rosendahl *et al.*,[6] we find large-avalanche tails in the distribution. In the case of large piles, the direct measurement of the mass fluctuations, i.e., neglecting the smaller avalanches, would lead to similar results as Jaeger *et. al.* found.

For the waiting-time distribution of the large avalanches, i.e., for the distribution of the number of dropping events between two consecutive large avalanches, surprisingly we find a double peak. Figure 4 shows the distribution for five different sizes of the discretization intervals. The double-peak structure is found in all five curves; hence we assume that it is not an artifact due to the choice of the size of the discretization interval. So far we have no explanation for this behavior.

Although our simulation seems at least not to contradict the concept of SOC, we should remark here that there are other serious objections against applying the idea of SOC in the case of sandpile avalanches (see, e.g., ref. 11).

ACKNOWLEDGMENTS

We thank the members of the Institute for their patience with regard to inconveniences connected with the permanent running on all computers of the job "HaufenGross." T.P. thanks H. Jaeger, H. J. Herrmann, and W. Ebeling for discussion. V.B. was supported by the Deutsche Forschungsgemeinschaft (RO 548/5-1).

REFERENCES

1. P. Bak, C. Tang, and K. Wiesenfeld, *Phys. Rev. Lett.* **59**:381 (1987); *Phys. Rev. A.* **38**:364 (1988); C. Tang and P. Bak, *J. Stat. Phys.* **51**:797 (1988).
2. K. Wiesenfeld, C. Tang, and P. Bak, *J. Stat. Phys.* **54**:1441 (1989), T. Hwa and M. Kardar, *Phys. Rev. Lett.* **62**:1813 (1989); D. Dhar, *Phys. Rev. Lett.* **64**:1613 (1990); B. McNamara and K. Wiesenfeld, *Phys. Rev. A* **41**:1867 (1990); H. Puhl, Self organized criticality. Ein Beispiel: Sandhaufen, Master's thesis, RWTH Aachen, Institut für Theoretische Physik und Forschungszentrum Jülich, HLRZ (1992).
3. V. Frette, *Phys. Rev. Lett.* **70**:2762 (1993), V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang, and P. Meakin, *Nature* **379**:49 (1996).
4. J. Kertész and L. Kiss, *J. Phys. A* **23**:L433 (1990); D. Dhar and R. Ramaswami, *Phys. Rev. Lett.* **63**:1659 (1990); S. N. Majumdar and D. Dhar, *J. Phys. A* **24**:L357 (1991); A. Vespignani, S. Zapperi, and L. Pietronero, *Phys. Rev. E* **51**:1711 (1995).
5. L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, *Phys. Rev. A* **39**:6524 (1989); K. Wiesenfeld, J. Theiler, and B. McNamara, *Phys. Rev. Lett.* **65**:949 (1990); G. A. Held, D. H. Solina, D. T. Keane, W. J. Haag, P. M. Horn, and G. Grinstein, *Phys. Lett.* **65**:1120 (1990); S. R. Nagel, *Rev. Mod. Phys.* **64**:321 (1992).
6. J. Rosendahl, M. Vekić, and J. Kelley, *Phys. Rev. E* **47**:1401 (1993).
7. H. M. Jaeger, C. Liu, and S. R. Nagel, *Phys. Rev. Lett.* **62**:40 (1989).
8. M. Bretz, J. B. Cunningham, P. L. Kurczynski, and F. Nori, *Phys. Rev. Lett.* **69**:2431 (1992).
9. T. Pöschel and V. Buchholtz, *Phys. Rev. Lett.* **71**:3963 (1993).
10. V. Buchholtz and T. Pöschel, *Physica A* **202**:390 (1994).
11. P. Evesque, *Europhys. Lett.* **14**:427 (1991); A. Mehta and G. C. Barker, *Europhys Lett.* **27**:501 (1994).

Communicated by D. Stauffer