

## Self-diffusion in granular gases

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The coefficient of self-diffusion for a homogeneously cooling granular gas changes significantly if the impact-velocity dependence of the restitution coefficient  $\epsilon$  is taken into account. For the case of a constant  $\epsilon$  the particles spread logarithmically slowly with time, whereas a velocity-dependent coefficient yields a power law time dependence. The impact of the difference in these time dependences on the properties of a freely cooling granular gas is discussed.

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### I. INTRODUCTION

The behavior of a freely evolving granular gas has been intensively discussed recently. In particular, the process of cluster formation due to inelastic collisions has been of wide interest, e.g., [1–3]. On the basis of a continuum description the effect of clustering in a force-free granular gas has been explained as an instability of the hydrodynamic equations [1–3]. For a deeper understanding of clustering phenomena it may be worth considering the processes in a granular gas that *precede* clustering. To this end we investigate the effect of self-diffusion of particles in the regime of homogeneous cooling.

The collisions of particles in a granular gas can be described by the coefficient of restitution  $\epsilon$  which relates the normal component of the relative velocity  $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$  before a collision to that after the collision  $\vec{v}'_{ij}$  as  $|\vec{v}'_{ij} \cdot \vec{e}| = \epsilon |\vec{v}_{ij} \cdot \vec{e}|$ . The unit vector  $\vec{e} = \vec{r}_{ij} / |\vec{r}_{ij}|$  gives the direction of  $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$  at the instant of the collision.

To our knowledge, all analytical calculations for the force-free case reported so far refer to systems of particles colliding with a constant restitution coefficient  $\epsilon$ . Experiments as well as theoretical studies show, however, that  $\epsilon$  depends on the normal component of the impact velocity  $|\vec{v}_{ij} \cdot \vec{e}|$  [4].

The problem of the restitution coefficient's dependence on the impact velocity has been addressed in [5], where the generalization of the Hertz contact problem was developed for the collision of three-dimensional (3D) viscoelastic spheres. From this generalized Hertz equation one obtains the velocity-dependent restitution coefficient [6] as an expansion:

$$\epsilon = 1 - C_1 \left( \frac{3A}{2} \right) \alpha^{2/5} |\vec{e} \cdot \vec{v}_{ij}|^{1/5} + C_2 \left( \frac{3A}{2} \right)^2 \alpha^{4/5} |\vec{e} \cdot \vec{v}_{ij}|^{2/5} \mp \dots \quad (1)$$

with

$$\alpha = \frac{2 Y \sqrt{R^{eff}}}{3 m^{eff} (1 - \nu^2)}, \quad (2)$$

where  $Y$  is the Young modulus,  $\nu$  is the Poisson ratio, and  $A$  depends on dissipative parameters of the particle material (for details see [5]). The effective mass and radius are defined as

$$R^{eff} = R_1 R_2 / (R_1 + R_2), \quad (3)$$

$$m^{eff} = m_1 m_2 / (m_1 + m_2), \quad (4)$$

with  $R_{1/2}$  and  $m_{1/2}$  being the radii and masses of the colliding particles. The constants  $C_1 = 1.15344$  and  $C_2 = 0.79826$  were obtained analytically [6] and then confirmed by numerical simulations and may be also written in a closed form as [7]

$$C_1 = \frac{\Gamma(3/5) \sqrt{\pi}}{2^{1/5} 5^{2/5} \Gamma(21/10)}, \quad (5)$$

$$C_2 = \frac{3}{5} C_1^2. \quad (6)$$

Equation (1) refers to the case of pure viscoelastic interaction, i.e., when the relative velocities  $|\vec{v}_{ij} \cdot \vec{e}|$  are not too large (to avoid plastic deformation of the particles) and not too small (to allow neglect of surface effects such as roughness, adhesion, and van der Waals interactions). This implies that the initial temperature of the granular gas is not too large and the final temperature is not too small. The range of validity of Eq. (1) depends on material and surface properties. In spite of such restrictions some important systems in nature do exist (e.g., planetary rings) where these conditions are satisfied [8]. Here we assume that the granular gas conditions allow for the application of Eq. (1).

For an equilibrium 3D system the time dependence of the mean-square displacement reads

$$\langle [\Delta r(t)]^2 \rangle_{eq} = 6Dt, \quad (7)$$

where  $\langle \dots \rangle_{eq}$  denotes the *equilibrium* ensemble averaging. To calculate the mean-square displacement, one writes

$$\langle [\Delta r(t)]^2 \rangle_{eq} = \left\langle \int_0^t \vec{v}(t') dt' \int_0^t \vec{v}(t'') dt'' \right\rangle_{eq} \quad (8)$$

and encounters then the velocity autocorrelation function,  $\langle \vec{v}(t') \cdot \vec{v}(t'') \rangle_{\text{eq}}$ . For systems at equilibrium this depends only on the time difference  $|t' - t''|$  and decays with a characteristic time  $\tau_v$ . Therefore, at time  $t \gg \tau_v$  one arrives at the self-diffusion coefficient

$$D = \frac{1}{3} \int_0^\infty \langle \vec{v}(0) \cdot \vec{v}(t) \rangle_{\text{eq}} dt. \quad (9)$$

For *nonequilibrium* systems such as granular materials the concept of the self-diffusion coefficient may also be applied, but with some care and with necessary generalization. Obviously, this refers only to “liquid” or gaseous phases of the material [9] where the particles have a noticeable mobility. In the following we also restrict ourselves to homogeneous cooling and consider the stages of evolution preceding the cluster formation [1,3], i.e., we assume that the granular material is (at least locally) homogeneous and isotropic. Hence, one can define the temperature  $T(t)$ , which decreases with time due to the loss of energy according to inelastic collisions. For the impact-velocity-dependent restitution coefficient (1) one has [6]

$$T = T_0 / (1 + t/\tau_0)^{5/3} \quad (10)$$

where  $T_0$  is the initial temperature and  $\tau_0$  is the characteristic time of the cooling process, which may be estimated as

$$\tau_0^{-1} \sim \sigma^2 n A \alpha^{2/5} T_0^{3/5} \quad (11)$$

with  $\sigma = 2R$  and  $n$  being the particle diameter and the particle number density, respectively. The mean collision time

$$\tau_c^{-1}(t) \equiv 4 \pi^{1/2} g_2(\sigma) n \sigma^2 T^{1/2} \quad (12)$$

depends on time via the time-dependent temperature. Here  $g_2(\sigma)$  is the contact value of the two-particle radial distribution function and the particles are of unit mass. Thus, the ratio of the two characteristic times reads

$$\tau_c(t)/\tau_0 \sim \delta^{11/6} [t/\tau_c(0)]^{5/6} \quad (13)$$

where  $\tau_c(0)^{-1}$  is the collisional frequency at initial time, and  $\delta = A \alpha^{2/5} T_0^{1/10}$  is supposed to be small. Clearly, one can employ the concept of temperature if  $\tau_c(t)/\tau_0 \ll 1$ . Thus, Eq. (13) gives the estimate  $t \ll \tau_c(0) \delta^{-11/5}$  for the upper time limit for which the use of the local temperature is justified.

An important property of a homogeneously cooling granular gas is that the velocity distribution is close to Maxwellian. Moreover, it persists with time, changing in accordance with the decreasing temperature [9]. The small value of the fourth cumulant of the velocity distribution function for any value of the restitution coefficient reported in [10] also supports the Maxwellian distribution [10,11]. Therefore, we assume that the Maxwellian distribution and the molecular chaos hypothesis may be used with a good degree of accuracy [12].

The evolution on the hydrodynamic time scale may be described using the kinetic coefficients calculated on the short time scale  $t \sim \tau_c$ . For granular gases these transport coefficient will be time dependent. We calculate the self-diffusion coefficient within the uncorrelated binary collisions

approximation and assume that the inequality  $\tau_c(t) \ll \tau_0$  always holds true (see the above discussion on this condition). Thus, on the time scale  $t \sim \tau_c$  the temperature may be considered as a constant. For  $t \gg \tau_c$ , however, the self-diffusion coefficient becomes time dependent (so that one prefers to call this quantity “diffusivity”) and the generalization of Eq. (7) reads

$$\langle [\Delta r(t)]^2 \rangle = 6 \int_0^t D(t') dt' \quad (14)$$

where  $\langle \dots \rangle$  denotes averaging over the *nonequilibrium* ensemble, whose evolution is described by a time-dependent  $N$ -particle distribution function  $\rho(t)$ . Here we restrict ourselves to times when the hydrodynamic contribution to the self-diffusion coefficient is not large [13], so that Eq. (14) with  $D(t)$  calculated on the time scale  $t \sim \tau_c$  gives an accurate description of the mean-square displacement.

The aim of the present study is to analyze how the velocity dependence of the restitution coefficient influences the diffusion in a gas of identical particles. The paper is organized as follows. In Sec. II we obtain the time dependence of the diffusivity and the temperature of the granular gas in a homogeneous cooling state. We also show that the mean-square displacement depends on time quite differently for the case of the constant restitution coefficient and for that determined by the impact velocity. In Sec. III we discuss our results for the mean-square displacement in the context of its possible impact on clustering.

## II. TIME DEPENDENCE OF THE DIFFUSION COEFFICIENT AND OF TEMPERATURE

To describe the dynamics of the granular material we use the formalism of the pseudo-Liouville operator  $\mathcal{L}$  [15]

$$i\mathcal{L} = \sum_j \vec{v}_j \cdot \frac{\partial}{\partial \vec{r}_j} + \sum_{i < j} \hat{T}_{ij}. \quad (15)$$

The first sum in Eq. (15) refers to the free streaming of the particles (the ideal part) while the second sum refers to the particle interactions, which are described by the binary collision operators [18]

$$\hat{T}_{ij} = \sigma^2 \int d^2 \vec{e} \Theta(-\vec{v}_{ij} \cdot \vec{e}) |\vec{v}_{ij} \cdot \vec{e}| \delta(\vec{r}_{ij} - \sigma \vec{e}) (\hat{b}_{ij}^{\vec{e}} - 1) \quad (16)$$

where  $\Theta(x)$  is the Heaviside function. The operator  $\hat{b}_{ij}^{\vec{e}}$  is defined as

$$\hat{b}_{ij}^{\vec{e}} f(\vec{r}_i, \vec{r}_j, \vec{v}_i, \vec{v}_j, \dots) = f(\vec{r}_i, \vec{r}_j, \vec{v}'_i, \vec{v}'_j, \dots), \quad (17)$$

where  $f$  is some function of dynamical variables. The after-collision velocities of the colliding particles,  $\vec{v}'_i$  and  $\vec{v}'_j$ , are related to their precollisional values  $\vec{v}_i$ ,  $\vec{v}_j$  via

$$\vec{v}'_{i,j} = \vec{v}_{i,j} + \frac{1}{2} (1 + \epsilon) (\vec{v}_{ij} \cdot \vec{e}) \vec{e}. \quad (18)$$

The pseudo-Liouville operator gives the time derivative of any dynamical variable  $B$  (e.g., [17]):

$$\frac{d}{dt}B(\{\vec{r}_i, \vec{v}_i\}, t) = i\mathcal{L}B(\{\vec{r}_i, \vec{v}_i\}, t) \quad (19)$$

and, therefore, the time evolution of  $B$  reads ( $t > t'$ )

$$B(\{\vec{r}_i, \vec{v}_i\}, t) = e^{i\mathcal{L}(t-t')}B(\{\vec{r}_i, \vec{v}_i\}, t'). \quad (20)$$

With Eq. (20) the time correlation function reads

$$\langle B(t')B(t) \rangle = \int d\Gamma \rho(t')B(t')e^{i\mathcal{L}(t-t')}B(t'), \quad (21)$$

where  $\int d\Gamma$  denotes integration over all degrees of freedom and  $\rho(t')$  depends on temperature  $T$ , density  $n$ , etc., which change on a time scale  $t \gg \tau_c$ . In accordance with the molecular chaos assumption at  $t \sim \tau_c$  the sequence of successive collisions occurs without correlations. If the variable  $B$  does not depend on the positions of the particles, its time correlation function reads [19]

$$\langle B(t')B(t) \rangle = \langle B^2 \rangle_{t'} e^{-|t-t'|/\tau_B(t')} \quad (t > t'), \quad (22)$$

where  $\langle \dots \rangle_{t'}$  denotes averaging with the distribution function taken at time  $t'$ . The relaxation time  $\tau_B$  is inversely proportional to the initial slope of the autocorrelation function [19]. It may be found from the time derivative of  $\langle B(t')B(t) \rangle$  taken at  $t = t'$ . Equations (21) and (22) then yield

$$-\tau_B^{-1}(t') = \int d\Gamma \rho(t') B i\mathcal{L}B / \langle B^2 \rangle_{t'} = \frac{\langle B i\mathcal{L}B \rangle_{t'}}{\langle B^2 \rangle_{t'}}. \quad (23)$$

The relaxation time  $\tau_B^{-1}(t')$ , depending on time via the distribution function  $\rho(t')$ , changes on the time scale  $t \gg \tau_c$ .

Let  $B(t)$  be the velocity of some particle, say  $\vec{v}_1(t)$ . Then with  $3T(t) = \langle v^2 \rangle_t$  Eqs. (22) and (23) [with Eqs. (15) and (16)] read

$$\langle \vec{v}_1(t') \cdot \vec{v}_1(t) \rangle = 3T(t') e^{-|t-t'|/\tau_v(t')}, \quad (24)$$

$$-\tau_v^{-1}(t') = (N-1) \frac{\langle \vec{v}_1 \cdot \hat{T}_{12} \vec{v}_1 \rangle_{t'}}{\langle \vec{v}_1 \cdot \vec{v}_1 \rangle_{t'}}. \quad (25)$$

To obtain Eq. (25) we take into account that  $\mathcal{L}_0 \vec{v}_1 = 0$ ,  $\hat{T}_{ij} \vec{v}_1 = 0$  if  $i \neq 1$  and the identity of the particles. The calculation of  $\tau_v^{-1}(t')$  may be performed if we assume that the distribution function  $\rho(t')$  is a product of the coordinate part, which corresponds to a uniform and isotropic system, and a velocity part that is a product of Maxwellian distribution functions,

$$\phi(\vec{v}_i) = \frac{\exp[-v_i^2/2T(t')]}{[2\pi T(t')]^{3/2}}, \quad i = 1, \dots, N. \quad (26)$$

Integration over the coordinate part in Eq. (25) yields

$$(N-1) \int \rho(t') \delta(\vec{r}_{ij} - \sigma \vec{e}) d\vec{r}_1 \dots d\vec{r}_N = n g_2(\sigma) \prod_i \phi(\vec{v}_i), \quad (27)$$

where we use the definition of the configurational distribution functions [17], and where [17,19]

$$g_2(\sigma) = \frac{1}{2}(2 - \eta)/(1 - \eta)^3 \quad (28)$$

gives the contact value of the configurational distribution function and  $\eta = \frac{1}{6}\pi n \sigma^3$ . With

$$\langle \vec{v}_1 \hat{T}_{12} \vec{v}_1 \rangle_{t'} = \frac{1}{2} \langle \vec{v}_{12} \hat{T}_{12} \vec{v}_1 \rangle_{t'} \quad (29)$$

due to the collision rules and definition (16), one finally arrives at

$$\tau_v^{-1}(t') = \frac{1}{4} n g_2(\sigma) \sigma^2 \int d\vec{v}_{12} \phi(\vec{v}_{12}) \quad (30)$$

$$\int d^2\vec{e} \Theta(-\vec{v}_{12} \cdot \vec{e}) |\vec{v}_{12} \cdot \vec{e}| (\vec{v}_{12} \cdot \vec{e})^2 (1 + \epsilon) / \langle \vec{v}_1 \cdot \vec{v}_1 \rangle_{t'},$$

where

$$\phi(\vec{v}_{12}) = (4\pi T)^{-3/2} \exp(-v_{12}^2/4T) \quad (31)$$

is the Maxwellian distribution for the relative velocity of two particles. For  $\epsilon$  not depending on  $v_{12}$ , Eq. (30) yields

$$\tau_v^{-1}(t) = \frac{\epsilon + 1}{2} \frac{8}{3} n \sigma^2 g_2(\sigma) \sqrt{\pi T(t)} = \frac{\epsilon + 1}{2} \tau_E^{-1}(t), \quad (32)$$

where  $\tau_E(t) = \frac{3}{2} \tau_c(t)$  is the Enskog relaxation time [17]. For the granular gas it depends on time according to the same time scale as the temperature. As shown in Eq. (32) the velocity correlation time for inelastic collisions exceeds that of elastic ones. This follows from partial suppression of the backscattering of particles due to inelastic losses in their normal relative motion. As a result the trajectories of particles are more stretched as compared with the elastic case, and therefore the velocity correlation time is larger.

As discussed above, a constant restitution coefficient is not consistent with the nature of the inelastic collisions. Substituting Eq. (1) into Eq. (30) one finds the velocity correlation time for the gas of inelastically colliding spheres:

$$\tau_v^{-1}(t) = \tau_E^{-1}(t) \left\{ 1 - \frac{3}{4} \Gamma\left(\frac{21}{10}\right) C_1 A \alpha^{2/5} [4T(t)]^{1/10} + \frac{27}{40} \Gamma\left(\frac{11}{5}\right) C_1^2 A^2 \alpha^{4/5} [4T(t)]^{1/5} + \dots \right\}, \quad (33)$$

where  $\Gamma(x)$  is the gamma function,  $\tau_E$  is given by Eq. (32) [21] and we use Eq. (6), which relates the coefficients  $C_1$  and  $C_2$ . From Eq. (33) it follows that the velocity autocorrelation function decays (as expected) on a short time scale, since  $\tau_v$  is of the order of  $\tau_c$ .

Using the velocity correlation function one writes

$$\langle [\Delta r(t)]^2 \rangle = 2 \int_0^t dt' 3T(t') \int_{t'}^t dt'' e^{-|t''-t'|/\tau_v(t')}. \quad (34)$$

On the short time scale  $t \sim \tau_c$ ,  $T(t')$  and  $\tau_v(t')$  may be considered as constants. Integrating in Eq. (34) over  $t''$  and equating with Eq. (14) yields for  $t \gg \tau_c \sim \tau_v$  the diffusivity (time-dependent self-diffusion coefficient)

$$D(t) = T(t) \tau_v(t). \quad (35)$$

Using the pseudo-Liouville operator one can also describe the time dependence of the temperature of the granular gas with the impact-velocity-dependent restitution coefficient. From Eq. (19) it follows that (see also [20])

$$\dot{T}(t) = \frac{1}{3} \frac{d}{dt} \langle v^2 \rangle_t = \frac{1}{3} \langle i\mathcal{L} v^2 \rangle_t. \quad (36)$$

Calculations similar to that for  $\tau_v(t')$  yield

$$\dot{T} = -b_1 T^{8/5} + b_2 T^{17/10} + \dots, \quad (37)$$

$$b_1 = 4 \times 2^{1/5} \pi^{1/2} \Gamma\left(\frac{21}{10}\right) C_1 \sigma^2 n g_2(\sigma) A \alpha^{2/5},$$

$$b_2 = \frac{33}{5} 2^{2/5} \pi^{1/2} \Gamma\left(\frac{11}{5}\right) C_1^2 \sigma^2 n g_2(\sigma) A^2 \alpha^{4/5}.$$

Solving Eq. (37) and expanding the result in terms of the small parameter

$$\delta = A \alpha^{2/5} T_0^{1/10}, \quad (38)$$

one arrives at

$$\frac{T(t)}{T_0} = \left(1 + \frac{t}{\tau_0}\right)^{-5/3} + a_1 \delta \left(1 + \frac{t}{\tau_0}\right)^{-11/6} + a_2 \delta^2 \left(1 + \frac{t}{\tau_0}\right)^{-2} + \dots \quad (39)$$

with  $a_1$  and  $a_2$  being pure numbers [22] and with

$$\tau_0^{-1} = \delta \tau_c(0)^{-1} \times \frac{3}{5} 2^{1/5} C_1 \Gamma\left(\frac{21}{10}\right) = 0.831928 \delta \tau_c(0)^{-1}. \quad (40)$$

The leading term in this expansion corresponds to the dependence (10) obtained previously using scaling arguments [6]. From Eqs. (35), (39), and (33) follows the time dependence of the self-diffusion coefficient:

$$\frac{D(t)}{D_0} = \left(1 + \frac{t}{\tau_0}\right)^{-5/6} + a_3 \delta \left(1 + \frac{t}{\tau_0}\right)^{-1} + a_4 \delta^2 \left(1 + \frac{t}{\tau_0}\right)^{-7/6} + \dots \quad (41)$$

with pure numbers  $a_3$  and  $a_4$  [22] and with

$$D_0^{-1} = \frac{8}{3} \pi^{1/2} \sigma^2 g_2(\sigma) n T_0^{-1/2}. \quad (42)$$

Correspondingly, the mean-square displacement reads asymptotically at  $\tau_0 \ll t$

$$\langle [\Delta r(t)]^2 \rangle \sim t^{1/6} + a_3 \delta \ln t. \quad (43)$$

This dependence holds true for time

$$\tau_c(0) \delta^{-1} \ll t \ll \tau_c(0) \delta^{-11/5}, \quad (44)$$

where the first inequality follows from the condition  $\tau_0 \ll t$ , while the second one follows from the discussed condition  $\tau_c(t) \ll \tau_0$ . Note that Eqs. (43) and (44) imply that terms of the order  $\sim \delta^2$  are negligible, which requires that the temperature be sufficiently small and/or the time be sufficiently large [23]. For a constant restitution coefficient one obtains

$$T(t)/T_0 = [1 + \gamma_0 t / \tau_c(0)]^{-2}, \quad (45)$$

where  $\gamma_0 \equiv (1 - \epsilon^2)/6$  [1,14]. Thus using Eqs. (32) and (35) one obtains for the mean-square displacement in this case

$$\langle [\Delta r(t)]^2 \rangle \sim \ln t. \quad (46)$$

It follows from Eqs. (43) and (46) that the impact-velocity-dependent restitution coefficient (1) leads to a significant change of the long time behavior of the mean-square displacement of particles in laboratory time. Compared to its logarithmically weak dependence for a constant restitution coefficient, the impact-velocity dependence of the restitution coefficient (1) gives rise to a considerably faster increase of this quantity with time, according to a power law.

### III. RESULTS AND DISCUSSION

We studied the diffusion of particles in a homogeneously cooling granular gas. With the assumption of molecular chaos we calculated the velocity time correlation function and the diffusivity (the time-dependent self-diffusion coefficient). For an impact-velocity-dependent restitution coefficient we found a relation that expresses the diffusivity in terms of the material constants of particles and characteristics of the granular gas, such as temperature, density, etc.

In our calculations we used the restitution coefficient for viscoelastic collision of particles, which depends on the normal component of the impact velocity  $|\vec{v}_{ij} \cdot \vec{e}|$  as a series  $\Sigma |\vec{v}_{ij} \cdot \vec{e}|^\gamma$  with  $\gamma = 1/5, \gamma = 2/5, \dots$  [Eq. (1)]. Our approach can be extended to arbitrary exponents  $\gamma$ , which let the expansion (1) converge. Using these exponents one may possibly describe collisions with very large impact velocities (when plastic deformation occurs) or very small velocities (when the surface effects are important), provided fragmentation and/or coagulation are ignored.

For granular particles suffering viscoelastic collisions we found that the mean-square displacement grows with time as a power law  $\sim t^{1/6}$ , i.e., much faster than the logarithmic growth  $\sim \ln t$ , observed in granular gases with a constant restitution coefficient. It is worth noting that qualitatively this power law dependence (as well as the logarithmic one) simply follows from scaling arguments and the time dependence of temperature. Indeed, the average velocity scales as  $\bar{v} \sim T^{1/2}$ , and therefore as  $\sim t^{-1}$  for a constant restitution coefficient and as  $\sim t^{-5/6}$  for an impact-velocity-dependent coefficient. The diffusivity in a granular gas scales as  $D \sim l^2 / \tau_c$ , where  $l \sim \sigma^{-2} n^{-1}$  is the mean free path, which does not change with time (in the regime preceding clustering), and  $\tau_c \sim l / \bar{v}$  is the mean collision time. Thus,  $D \sim l \bar{v} \sim T^{1/2}$ , and we obtain that the mean-square displacement,  $\int^t D(t) dt$ , scales as  $\sim \ln t$  in the former case and as  $\sim t^{1/6}$  in the latter case.

What will be the impact of this apparently dramatic difference in the time dependence of  $\langle [\Delta r(t)]^2 \rangle$  on the properties of granular gases? In laboratory time this corresponds to the enhanced spreading (and therefore mixing) of particles with the velocity-dependent restitution coefficient as compared to the case of constant  $\epsilon$ . Since the temperature decreases more slowly for the former case, as  $\sim t^{-5/3}$ , as com-

pared with  $\sim t^{-2}$  for the latter one, retarded clustering may be expected.

One might wish to define the average cumulative number of collisions per particle  $\mathcal{N}(t)$  as a system-inherent time scale ( $t$  is the laboratory time) and compare dynamics of the systems in their inherent time scales.  $\mathcal{N}(t)$  is easily accessible in numerical simulations and is a convenient quantity to analyze evolution of granular gases (e.g., [24,26]). It may be found by integrating  $d\mathcal{N} = \tau_c(t)^{-1} dt$  [14]. For a constant restitution coefficient one obtains  $\mathcal{N}(t) \sim \ln t$ , while for the impact-velocity-dependent  $\mathcal{N}(t) \sim t^{1/6}$ . Therefore, the mean-square displacement behaves identically in both cases as

$$\langle [\Delta r(\mathcal{N})]^2 \rangle \sim \mathcal{N}. \quad (47)$$

If the number of collisions per particle  $\mathcal{N}(t)$  were the only quantity specifying the stage of the granular gas evolution, one would conjecture that the dynamical behavior of a granular gas with a constant  $\epsilon$  and a velocity-dependent  $\epsilon$  is identical provided the  $\mathcal{N}$ -based time scale is used. According to our understanding, however, this is not an adequate description of physical reality. Indeed, as was shown in Ref. [24], the values of  $\mathcal{N}_c$  corresponding to a crossover from the linear regime of evolution (which refers to homogeneous cooling) to the nonlinear regime (when clustering starts) may differ by orders of magnitude, depending on the restitution coefficient and on the density of the granular gas. Therefore, to analyze the behavior of a granular gas one can try an alternative inherent time scale,  $\mathcal{T}^{-1} \equiv T(t)/T_0$ , which is based on the gas temperature [25]. Given two systems of granular particles at the same density and the same initial temperature  $T_0$ , consisting of particles colliding with constant and velocity-dependent restitution coefficients, respectively, the time  $\mathcal{T}$  allows us to compare their evolution directly. A strong argument for using a temperature-based time has been given by Goldhirsch and Zanetti [1]. They found that there are two main contributions to the rate of temperature decay. The first refers to cooling due to inelastic collisions with a rate  $\sim T^{3/2}$  (for constant  $\epsilon$ ) and corresponds to homogeneous cooling. The second refers to viscous heating and behaves as  $\sim T^{1/2}$ . Initially, the first is much larger than the second, but when temperature decays, the second contribution takes over and the shear wave adiabatically enslaves the temperature field [1]. This corresponds to the nonlinear

stage of evolution, when clustering starts. Thus, the temperature may indicate the stage of evolution of a granular gas. The recent numerical results of Ref. [24] strongly support our assumption. It was shown that while  $\mathcal{N}_c$  differs by more than a factor of 3 for two different systems ( $\mathcal{N}_c = 70$  for a system with  $\epsilon = 0.9$  and packing fraction  $\phi = 0.245$ , and  $\mathcal{N}_c = 23$  for  $\epsilon = 0.6$  and  $\phi = 0.05$  [24]), the values of  $\mathcal{T}_c$  [defined as  $T(\mathcal{N}_c)/T_0$ ], are very close ( $\mathcal{T}_c \approx 0.0031$  for the first system and  $\mathcal{T}_c \approx 0.0027$  for the second [24]). These arguments show that one could consider  $\mathcal{T}$  as a relevant time scale to analyze granular gas evolution.

With  $T(\mathcal{N})/T_0 \sim e^{-2\gamma_0\mathcal{N}}$  for a constant restitution coefficient and  $T(\mathcal{N})/T_0 \sim \mathcal{N}^{-10}$  for an impact-velocity-dependent one, we obtain  $\langle (\Delta r)^2 \rangle \sim \ln \mathcal{T}$  for the case of constant  $\epsilon$  and

$$\langle (\Delta r)^2 \rangle \sim \mathcal{T}^{1/10} \quad (48)$$

for the velocity-dependent restitution coefficient. Thus, for the temperature-based inherent time scale we again obtain a power law dependence for the latter system and a logarithmically weak time-dependence for the former.

In conclusion, we found that the impact-velocity dependence of the restitution coefficient  $\epsilon$  significantly influences the mean-square displacement of the particles in a granular gas in laboratory time. As compared with the logarithmically weak time dependence found for a constant restitution coefficient, the impact-velocity-dependent coefficient (1) yields a power law (43). It causes increasingly enhanced spreading of the particles through the system. This together with the fact that the temperature decreases more slowly for the velocity-dependent  $\epsilon$  leads to the suggestion of retarded clustering in such systems.

We also analyzed the mean-square displacement of particles using different inherent time scales. We found that while for the number-of-collisions-based time scale both systems behave identically, for the temperature-based time scale power law behavior is still observed for the case of the velocity-dependent coefficient and logarithmically weak behavior for the case of constant  $\epsilon$ .

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