

## Velocity distribution in granular gases of viscoelastic particles

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The velocity distribution in a homogeneously cooling granular gas has been studied in the viscoelastic regime, when the restitution coefficient of colliding particles depends on the impact velocity. We show that for viscoelastic particles a simple scaling hypothesis is violated, i.e., that the time dependence of the velocity distribution does not scale with the mean square velocity as in the case of particles interacting via a constant restitution coefficient. The deviation from the Maxwellian distribution does not depend on time monotonically. For the case of small dissipation we detected two regimes of evolution of the velocity distribution function: Starting from the initial Maxwellian distribution, the deviation first increases with time on a collision time scale saturating at some maximal value; then it decays to zero on a much larger time scale which corresponds to the temperature relaxation. For larger values of the dissipation parameter there appears an additional intermediate relaxation regime. Analytical calculations for small dissipation agree well with the results of a numerical analysis.

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### I. INTRODUCTION

The statistical properties of granular gases have been intensively studied recently, in particular with respect to the cluster formation process [1] and other structure formation processes [2]. In the present paper we are concerned with the dynamical processes in granular gases which precede clustering, i.e., in the homogeneously cooling state (HCS). As opposed to the state when particles form clusters and other long range structures, in the HCS (due to its definition) one may drop the explicit spatial dependence of the statistical properties, which simplifies an application of standard methods of the gas kinetic gas theory. Granular gases in the HCS were intensively investigated recently (see, e.g., Ref. [3] for a review) focusing on the velocity distribution function which is one of the most important characteristics of the system of granular particles. It was argued that the distribution function might deviate from the Maxwellian [4,5], and this deviation was also quantified [4,6,7].

In all of these studies a constant restitution coefficient, characterizing energy loss due to a particle collision was assumed. The restitution coefficient relates the velocities of the colliding particles before a collision  $\vec{v}_1$  and  $\vec{v}_2$  to the velocities after the collision,  $\vec{v}_1^*$  and  $\vec{v}_2^*$ :

$$\vec{v}_1^* = \vec{v}_1 - \frac{1}{2}(1 + \epsilon)(\vec{v}_{12} \cdot \vec{e})\vec{e}, \quad (1)$$

$$\vec{v}_2^* = \vec{v}_2 + \frac{1}{2}(1 + \epsilon)(\vec{v}_{12} \cdot \vec{e})\vec{e},$$

where  $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$  is the relative velocity, and the unit vector  $\vec{e} = \vec{r}_{12}/|\vec{r}_{12}|$  gives the direction of the intercenter vector  $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$  at the instant of the collision. Strictly speaking the restitution coefficient  $\epsilon$ , as introduced in Eq. (1), describes the collision of *smooth* inelastic particles, when only

the normal component ( $\vec{v}_{12} \cdot \vec{e}$ ) of the relative velocity  $\vec{v}_{12}$  changes. Therefore, it is termed the *normal* restitution coefficient. Using the tangential restitution coefficient [9,8,10], one can account for the change in tangential component of the relative velocity at the collision of rough inelastic particles. In what follows we assume that the particles are smooth and that the dynamics of a collision is completely described by the change of the normal component of the relative velocity.

However, experiments, as well as theoretical studies showed that  $\epsilon$  noticeably depends on the impact velocity  $\vec{v}_{12}$  [12–15]; even a dimension analysis shows that the assumption of the constant restitution coefficient contradicts physical reality [16,17]. This dependence may cause rather important consequences for various problems in granular gas dynamics [18,19]. The problem of the restitution coefficient's dependence on the impact velocity was addressed in Refs. [8,11], where the generalization of the Hertz contact problem was developed for the collision of viscoelastic particles (a scaling analysis of this dependence was also addressed in Ref. [14]). The generalized Hertz collision equation derived in Ref. [8] was solved analytically to obtain the velocity-dependent restitution coefficient [20]

$$\epsilon = 1 - C_1 A \alpha^{2/5} |\vec{v}_{12} \cdot \vec{e}|^{1/5} + C_2 A^2 \alpha^{4/5} |\vec{v}_{12} \cdot \vec{e}|^{2/5} \mp \dots, \quad (2)$$

with

$$\alpha = \left(\frac{3}{2}\right)^{3/2} \frac{Y \sqrt{R^{\text{eff}}}}{m^{\text{eff}}(1 - \nu^2)}, \quad (3)$$

where  $Y$  is the Young modulus,  $\nu$  is the Poisson ratio,  $R^{\text{eff}} = R_1 R_2 / (R_1 + R_2)$ ,  $m^{\text{eff}} = m_1 m_2 / (m_1 + m_2)$  ( $R_{1/2}$  and  $m_{1/2}$  are radii and masses of colliding particles), and  $A$  is the dissipative constant, which depends on the material param-

eters (see Ref. [8] for details). Numerical values for the constants  $C_1$  and  $C_2$  obtained in Ref. [20] may be also written in more convenient forms [17]:

$$C_1 = \frac{\Gamma(3/5)\sqrt{\pi}}{2^{1/5}5^{2/5}\Gamma(21/10)} = 1.15344, \quad (4)$$

$$C_2 = \frac{3}{5}C_1^2. \quad (5)$$

Although the next-order coefficients  $C_3 = 0.315119C_1^3$  and  $C_4 = 0.161170C_1^4$ , are now available [17], we assume that the dissipative constant  $A$  is small enough to ignore these high-order terms.

The aim of the present study is to analyze how the impact-velocity-dependent restitution coefficient, given by Eq. (2) for the collision of viscoelastic spheres, influences the velocity distribution in a granular gas of identical particles in the HCS. To address this problem we use the Sonine polynomial expansion for the velocity distribution function, and analyze the time dependence of the expansion coefficients.

We want to mention that throughout this paper we assume viscoelastic particle deformation. This refers to relatively small impact velocities, and, as shown in Refs. [8,17], applies to conditions important for astrophysical systems [13]. Other regimes of deformation which are important for high impact velocities, such as plastic deformation or brittle fracture, have not been taken into account.

In Sec. II we introduce the necessary variables, briefly sketch the method of Sonine polynomial expansion, and summarize the knowledge about the velocity distribution function in granular gases under the assumption of a constant restitution coefficient. In Sec. III we analyze the Boltzmann equation for the granular gas with a velocity-dependent  $\epsilon$  in the HCS, and calculate the first few coefficients of the Sonine polynomials expansion. We show that these coefficients turn out to be time dependent, so that the velocity distribution function does not have a simple scaling form. In Sec. IV we consider the time evolution of the temperature and the velocity distribution. The high-velocity tail of the distribution function is analyzed in Sec. V. In Sec. VI we summarize our findings. Some technical detail of the calculations are given in the Appendixes.

## II. SONINE POLYNOMIAL EXPANSION FOR GRANULAR GASES

For granular gases where the particles interact via a restitution coefficient  $\epsilon = \text{const}$ , it was argued that the velocity distribution  $f(\vec{v}, t)$  has a scaling form, i.e., that its time-dependence may be written as (here we follow notations of Ref. [6])

$$f(\vec{v}, t) = \frac{n}{v_0^d(t)} \tilde{f}\left(\frac{\vec{v}}{v_0(t)}\right), \quad (6)$$

where  $n$  is the number density of the granular gas;  $v_0(t)$  is the thermal velocity, defined in terms of the temperature of the granular gas

$$T(t) = \frac{1}{2} m v_0^2(t); \quad (7)$$

$m$  is the mass of the granular particles; and  $d$  is the dimension. The temperature is related to the second moment of the velocity distribution in the same way as for equilibrium molecular systems:

$$\frac{d}{2} n T(t) = \int d\vec{v} \frac{m v^2}{2} f(\vec{v}, t). \quad (8)$$

Then the expansion of the scaling function  $\tilde{f}(\vec{c})$  [where  $\vec{c} \equiv \vec{v}/v_0(t)$ ] in terms of the Sonine polynomials reads [4,6]

$$\tilde{f}(\vec{c}) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p S_p(c^2) \right\}, \quad (9)$$

where  $\phi(c) \equiv \pi^{-d/2} \exp(-c^2)$  is the Maxwellian distribution for the rescaled velocity. The Sonine polynomials  $S_p(c^2)$  satisfy the orthogonality conditions

$$\int d\vec{c} \phi(c) S_p(c^2) S_{p'}(c^2) = \delta_{pp'} \mathcal{N}_p, \quad (10)$$

with  $\delta_{pp'}$  being the Kronecker  $\delta$ , and with the normalization constant  $\mathcal{N}_p$  [4,6]. For dimension  $d=3$ , which is addressed in the present study, the first few Sonine polynomials read

$$S_0(x) = 1,$$

$$S_1(x) = -x^2 + \frac{3}{2}, \quad (11)$$

$$S_2(x) = \frac{x^2}{2} - \frac{5x}{2} + \frac{15}{8}. \quad (12)$$

The coefficients  $a_p$  of the expansion may be found as the polynomial moments of the function  $\tilde{f}(\vec{c})$  [4,6]:

$$a_p = \frac{1}{\mathcal{N}_p} \int d\vec{c} S_p(c^2) \tilde{f}(\vec{c}). \quad (13)$$

The coefficients  $a_p$  do not depend on time for a *constant* restitution coefficient [21]. These were first applied for the granular gas in Ref. [4], and then recalculated recently [6]:

$$a_1 = 0, \quad (14)$$

$$a_2 = \frac{16(1-\epsilon)(1-2\epsilon^2)}{9+24d+8\epsilon d+41\epsilon+30(1-\epsilon)\epsilon^2}. \quad (15)$$

The first relation [Eq. (14)] follows from the definition of the temperature of the granular gas (this we explain in more detail below), while Eq. (15) has been obtained within the linear approximation with respect to  $a_2$ . A complete analysis, which goes beyond the linear approximation, was performed [7], and it was shown [7] that linear solution (15) is rather accurate for the whole range of  $\epsilon$  with a maximal deviation from a total one less than 10% [22]. All the higher-order coefficients were neglected under the assumption of small deviations from the Maxwellian distribution. Since  $a_p$

do not depend on time, the scaling form of the velocity distribution function (6) persists with time for the case of  $\epsilon = \text{const}$ .

Since the average velocity of a granular gas decreases due to the permanently decreasing temperature, the ‘‘typical’’ restitution coefficient will increase with time, as follows from Eq. (2). Thus one can expect that the coefficients of the Sonine polynomial expansion, which depend on the restitution coefficient [see, e.g., Eq. (15)] should also change with time. This conclusion, however, contradicts the assumption that the scaling function (9) does not depend on time, and implies that the common scheme of calculation of the Sonine polynomials expansion coefficients breaks down if  $\epsilon$  is not a constant. For the latter case, one needs to develop a more general approach.

### III. KINETIC EQUATION FOR THE COEFFICIENTS OF THE SONINE POLYNOMIAL EXPANSION

We start from the Enskog-Boltzmann equation for the distribution function  $f(\vec{r}, \vec{v}, t)$  for a granular gas of inelastic spheres, which in the force-free case does not depend on  $\vec{r}$ . Hence one can write [6,23]

$$\begin{aligned} \frac{\partial}{\partial t} f(\vec{v}_1, t) &= g_2(\sigma) \sigma^2 \int d\vec{v}_2 \int d\vec{e} \Theta(-\vec{v}_{12} \cdot \vec{e}) |\vec{v}_{12} \cdot \vec{e}| \\ &\times \{ \chi f(\vec{v}_1^{**}, t) f(\vec{v}_2^{**}, t) - f(\vec{v}_1, t) f(\vec{v}_2, t) \} \\ &\equiv g_2(\sigma) I(f, f), \end{aligned} \quad (16)$$

where  $\sigma$  is the diameter of the particles. The contact value of the two-particle correlation function,  $g_2(\sigma) = (2 - \eta)/2(1 - \eta)^3$  [24] (with  $\eta = \frac{1}{6} \pi n \sigma^3$  being the packing fraction), accounts for the increasing collision frequency due to the excluded volume effects.  $\Theta(x)$  is the Heaviside step function. The velocities  $\vec{v}_1^{**}$  and  $\vec{v}_2^{**}$  refer to the precollisional velocities of the so-called inverse collision, which results with  $\vec{v}_1$  and  $\vec{v}_2$  as the after-collisional velocities [the relation between these velocities are similar to that of Eq. (1), but with an impact-velocity-dependent restitution coefficient; see Appendix A]. Finally the factor

$$\chi = 1 + \frac{11}{5} C_1 A \alpha^{2/5} |\vec{v}_{12} \cdot \vec{e}|^{1/5} + \frac{66}{25} C_1^2 A^2 \alpha^{4/5} |\vec{v}_{12} \cdot \vec{e}|^{2/5} + \dots \quad (17)$$

in the gain term appears, respectively, from the Jacobian of the transformation  $d\vec{v}_1^{**} d\vec{v}_2^{**} \rightarrow d\vec{v}_1 d\vec{v}_2$  and from the relation between the lengths of the collisional cylinders  $\epsilon |\vec{v}_{12}^{**} \cdot \vec{e}| dt = |\vec{v}_{12} \cdot \vec{e}| dt$  (see Appendix A for details). For the constant restitution coefficient,  $\chi = 1/\epsilon^2 = \text{const}$ .

Some important properties of the collisional integral also hold for the case of the impact-velocity-dependent restitution coefficient. That is, it may be shown that the relation

$$\begin{aligned} \frac{d}{dt} \langle \psi(t) \rangle &= \int d\vec{v}_1 \psi(\vec{v}_1) \frac{\partial}{\partial t} f(\vec{v}_1, t) = \int d\vec{v}_1 \psi(\vec{v}_1) I(f, f) \\ &= \frac{g_2(\sigma) \sigma^2}{2} \int d\vec{v}_1 d\vec{v}_2 \int d\vec{e} \Theta(-\vec{v}_{12} \cdot \vec{e}) \\ &\times |\vec{v}_{12} \cdot \vec{e}| f(\vec{v}_1, t) f(\vec{v}_2, t) \Delta[\psi(\vec{v}_1) + \psi(\vec{v}_2)] \end{aligned} \quad (18)$$

holds true, where  $\langle \psi(t) \rangle \equiv \int d\vec{v}_1 \psi(\vec{v}) f(\vec{v}, t)$  is the average of some function  $\psi(\vec{v})$ , and  $\Delta \psi(\vec{v}_i) \equiv [\psi(\vec{v}_i^*) - \psi(\vec{v}_i)]$  denotes a change of  $\psi(\vec{v}_i)$  in a direct collision.

Now we analyze the scaling ansatz (6) for the velocity distribution function. Using this ansatz and performing calculations similar to that in Ref. [6], one would find corresponding expressions for the coefficients  $a_p$  of the Sonine polynomial expansion. These would turn out to be time dependent due to the permanently decreasing average velocity of the cooling gas and thus the permanently increasing effective value of the restitution coefficient. However, this means that the simple scaling (6) for the velocity distribution function does not hold for the case of interest. Technically, as we show below, this follows from the additional time dependence of the factor  $\chi$  in the collisional integral, which does not depend on time for  $\epsilon = \text{const}$ .

Thus it seems natural to write the three-dimensional distribution function in the general form

$$f(\vec{v}, t) = \frac{n}{v_0^3(t)} \tilde{f}(\vec{c}, t), \quad (19)$$

with

$$\tilde{f}(\vec{c}, t) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p(t) S_p(c^2) \right\}, \quad (20)$$

and then find equations for the *time-dependent* coefficients  $a_p(t)$ .

Substituting Eq. (19) into the Boltzmann equation (16), we obtain

$$\begin{aligned} \frac{1}{v_0^2} \frac{dv_0}{dt} \left( 3 + c_1 \frac{\partial}{\partial c_1} \right) \tilde{f}(\vec{c}_1, t) + \frac{1}{v_0} \frac{\partial}{\partial t} \tilde{f}(\vec{c}_1, t) \\ = g_2(\sigma) \sigma^2 n \tilde{I}(\tilde{f}, \tilde{f}), \end{aligned} \quad (21)$$

where we define the dimensionless collisional integral

$$\begin{aligned} \tilde{I}(\tilde{f}, \tilde{f}) &= \int d\vec{c}_2 \int d\vec{e} \Theta(-\vec{c}_{12} \cdot \vec{e}) |\vec{c}_{12} \cdot \vec{e}| \\ &\times \{ \tilde{\chi} \tilde{f}(\vec{c}_1^{**}, t) \tilde{f}(\vec{c}_2^{**}, t) - \tilde{f}(\vec{c}_1, t) \tilde{f}(\vec{c}_2, t) \}. \end{aligned} \quad (22)$$

The reduced factor  $\tilde{\chi}$ ,

$$\tilde{\chi} = 1 + \frac{11}{5} C_1 \delta' |\vec{c}_{12} \cdot \vec{e}|^{1/5} + \frac{66}{25} C_1^2 \delta'^2 |\vec{c}_{12} \cdot \vec{e}|^{2/5} + \dots, \quad (23)$$

depends now on time via a quantity

$$\delta'(t) \equiv A\alpha^{2/5}[2T(t)]^{1/10} \equiv \delta[2T(t)/T_0]^{1/10}. \quad (24)$$

Here  $\delta \equiv A\alpha^{2/5}(T_0)^{1/10}$ ,  $T_0$  is the initial temperature, and for simplicity we assume the particles to be of unit mass,  $m = 1$ .

The rate of change of the thermal velocity  $dv_0/dt$  in Eq. (21) may be expressed in terms of the temperature decay rate  $dT/dt$ , which reads, according to definitions (7) and (8) and relation (18) for the time derivatives of averages,

$$\frac{dT}{dt} = \frac{1}{3}g_2(\sigma)\sigma^2nv_0^3 \int d\vec{c}_1c_1^2\tilde{I}(\vec{f},\vec{f}) = -\frac{2}{3}BT\mu_2. \quad (25)$$

Here we define  $B = B(t) \equiv v_0(t)g_2(\sigma)\sigma^2n$ , and introduce the moments of the dimensionless collision integral:

$$\mu_p \equiv - \int d\vec{c}_1c_1^p\tilde{I}(\vec{f},\vec{f}). \quad (26)$$

With this notation we recast Eq. (21) into the form

$$\frac{\mu_2}{3} \left( 3 + c_1 \frac{\partial}{\partial c_1} \right) \tilde{f}(\vec{c}, t) + B^{-1} \frac{\partial}{\partial t} \tilde{f}(\vec{c}, t) = \tilde{I}(\vec{f}, \vec{f}). \quad (27)$$

Note that contrary to the case of  $\epsilon = \text{const}$ , where  $\chi = 1/\epsilon^2 = \text{const}$ , the factor  $\chi$  now depends on time, which does not allow one to write the collision integral in terms of only scaling variables. This implies a time dependence of all the moments  $\mu_p$  (which were time independent for the constant restitution coefficient), and correspondingly causes a time dependence of the Sonine polynomials expansion coefficients  $a_p$ .

Multiplying both sides of Eq. (27) by  $c_1^p$ , and integrating over  $\vec{c}_1$ , we obtain

$$\frac{\mu_2}{3} p \langle c^p \rangle - B^{-1} \sum_{k=1}^{\infty} \dot{a}_k \nu_{kp} = \mu_p, \quad (28)$$

where integration by parts has been performed, and where we define

$$\nu_{kp} \equiv \int \phi(c) c^p S_k(c^2) d\vec{c} \quad (29)$$

and

$$\langle c^p \rangle \equiv \int c^p \tilde{f}(\vec{c}, t) d\vec{c}. \quad (30)$$

The calculation of  $\nu_{kp}$  is straightforward; the first of them read

$$\nu_{22} = 0, \quad \nu_{24} = \frac{15}{4}. \quad (31)$$

The odd moments  $\langle c^{2n+1} \rangle$  are zero, while the even ones  $\langle c^{2n} \rangle$  may be expressed in terms of  $a_k$  with  $0 \leq k \leq n$ . This follows from the fact that  $c^{2n}$  may be written as a sum of Sonine polynomials  $S_k(c^2)$ , with  $0 \leq k \leq n$ , and from the or-

thogonality condition (10). That is, using  $c^2 = \frac{3}{2}S_0(c^2) - S_1(c^2)$  together with expansion (20) and condition (10), one easily finds

$$\begin{aligned} \langle c^2 \rangle &= \int d\vec{c} \phi(c) \left[ \frac{3}{2}S_0(c^2) - S_1(c^2) \right] \left\{ \sum_{k=0}^{\infty} a_k S_k(c^2) \right\} \\ &= \frac{3}{2} - \frac{3}{2}a_1, \end{aligned} \quad (32)$$

with  $a_0 = 1$ , and where we use the normalization constant  $\mathcal{N}_1 = \frac{3}{2}$  [see Eq. (10)]. From the definitions of temperature and of the thermal velocity [Eqs. (7) and (8)] it follows that  $\langle c^2 \rangle = \frac{3}{2}$  [also see Eq. 30]. Then Eq. (32) implies  $a_1 = 0$  in accordance with Ref. [6]. Similar considerations yield

$$\langle c^4 \rangle = \frac{15}{4}(1 + a_2). \quad (33)$$

The moments  $\mu_p$  may be also expressed in terms of coefficients  $a_2, a_3, \dots$ ; therefore, system (28) is an infinite (but closed) set of equations for these coefficients.

It is not possible to obtain a general solution to the problem. However, since the dissipative parameter  $\delta$  is supposed to be small, the deviations from the Maxwellian distribution are not presumably large. Thus we assume that one can neglect all the high-order terms in the expansion (20) with  $p > 2$ . Then Eq. (28) is an equation for the coefficient  $a_2$ . For  $p = 2$ , Eq. (28) converts into an identity, since  $\langle c^2 \rangle = \frac{3}{2}$ ,  $a_1 = 0$ , and due to Eq. (31). For  $p = 4$ , we obtain

$$\dot{a}_2 - \frac{4}{3}B\mu_2(1 + a_2) + \frac{4}{15}B\mu_4 = 0, \quad (34)$$

where relations (31) and (33) have been used. In Eq. (34),  $B$  depends on time as

$$B(t) = (8\pi)^{-1/2} \tau_c(0)^{-1} [T(t)/T_0]^{1/2}, \quad (35)$$

where  $T_0$  is the initial temperature and  $\tau_c(0)$  is the initial mean-collision time

$$\tau_c(0)^{-1} = 4\pi^{1/2}g_2(\sigma)\sigma^2nT_0^{1/2}. \quad (36)$$

The time evolution of the temperature is determined by Eq. (25), i.e., by the time dependence of  $\mu_2$ .

The time-dependent coefficients  $\mu_p(t)$  may be expressed in terms of  $a_2$  according to definition (26) and the approximation  $\tilde{f} = \phi(c)[1 + a_2(t)S_2(c^2)]$ . One obtains

$$\begin{aligned} \mu_p &= -\frac{1}{2} \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e} \Theta(-\vec{c}_{12} \cdot \vec{e}) |\vec{c}_{12} \cdot \vec{e}| \phi(c_1) \phi(c_2) \\ &\quad \times \{ 1 + a_2 [S_2(c_1^2) + S_2(c_2^2)] \\ &\quad + a_2^2 S_2(c_1^2) S_2(c_2^2) \} \Delta(c_1^p + c_2^p), \end{aligned} \quad (37)$$

with the definition of  $\Delta(c_1^p + c_2^p)$  given above. After long and tedious calculations (details are given in Appendix B), one arrives at the following result for the moments:

$$\mu_2 = \delta' [\mathcal{A}_1 + \mathcal{A}_2 a_2 + \mathcal{A}_3 a_2^2] - \delta'^2 [\mathcal{A}_4 + \mathcal{A}_5 a_2 + \mathcal{A}_6 a_2^2] \quad (38)$$

and

$$\begin{aligned} \mu_4 = & [\mathcal{B}_1 + \mathcal{B}_2 a_2 + \mathcal{B}_3 a_2^2] + \delta' [\mathcal{B}_4 + \mathcal{B}_5 a_2 + \mathcal{B}_6 a_2^2] \\ & - \delta'^2 [\mathcal{B}_7 + \mathcal{B}_8 a_2 + \mathcal{B}_9 a_2^2], \end{aligned} \quad (39)$$

where  $\mathcal{A}_n$  and  $\mathcal{B}_n$  are pure numbers. The coefficients  $\mathcal{A}_n$  read

$$\begin{aligned} \mathcal{A}_1 = \omega_0, \quad \mathcal{A}_2 = \frac{6}{25} \omega_0, \quad \mathcal{A}_3 = \frac{21}{2500} \omega_0, \\ \mathcal{A}_4 = \omega_1, \quad \mathcal{A}_5 = \frac{119}{400} \omega_1, \quad \mathcal{A}_6 = \frac{4641}{640000} \omega_1, \end{aligned} \quad (40)$$

with

$$\omega_0 \equiv 2\sqrt{2}\pi 2^{1/10} \Gamma\left(\frac{21}{10}\right) C_1 = 6.48562 \dots, \quad (41)$$

$$\omega_1 \equiv \sqrt{2}\pi 2^{1/5} \Gamma\left(\frac{16}{5}\right) C_1^2 = 9.28569 \dots, \quad (42)$$

and the coefficients  $\mathcal{B}_n$  are

$$\begin{aligned} \mathcal{B}_1 = 0, \quad \mathcal{B}_2 = 4\sqrt{2}\pi, \quad \mathcal{B}_3 = \frac{1}{8}\sqrt{2}\pi, \\ \mathcal{B}_4 = \frac{56}{10}\omega_0, \quad \mathcal{B}_5 = \frac{1806}{250}\omega_0, \quad \mathcal{B}_6 = \frac{567}{12500}\omega_0, \\ \mathcal{B}_7 = \frac{77}{10}\omega_1, \quad \mathcal{B}_8 = \frac{149054}{13750}\omega_1, \quad \mathcal{B}_9 = \frac{348424}{5500000}\omega_1. \end{aligned} \quad (43)$$

Thus Eqs. (34) and (25), together with Eqs. (24), (35), (38), and (39), form a closed set to find the time evolution of the temperature and the coefficient  $a_2$ . We want to stress an important difference for the time evolution of the temperature for the case of the impact-velocity-dependent restitution coefficient, compared to that of the constant restitution coefficient. In the former case it is coupled to the time evolution of the coefficient  $a_2$ , while in the latter case there is no such coupling since  $a_2 = \text{const}$ . This coupling may lead in some cases to a rather peculiar time dependence of the temperature. The problem of the time dependence of the temperature and the velocity distribution function will be discussed in detail in Sec. IV.

#### IV. TIME EVOLUTION OF TEMPERATURE AND THE VELOCITY DISTRIBUTION FUNCTION

To analyze the time evolution of the temperature and of the coefficient  $a_2$ , characterizing the velocity distribution function, we introduce the reduced temperature  $u(t) \equiv T(t)/T_0$ , and recast the set of equations (34) and (25) into the forms

$$\begin{aligned} \dot{u} + \tau_0^{-1} u^{8/5} \left( \frac{5}{3} + \frac{2}{5} a_2 + \frac{7}{500} a_2^2 \right) \\ - \tau_0^{-1} q_1 \delta u^{17/10} \left( \frac{5}{3} + \frac{119}{240} a_2 + \frac{1547}{128000} a_2^2 \right) = 0, \end{aligned} \quad (44)$$

$$\dot{a}_2 - r_0 u^{1/2} \mu_2 (1 + a_2) + \frac{1}{5} r_0 u^{1/2} \mu_4 = 0. \quad (45)$$

The characteristic time

$$\tau_0^{-1} = \frac{16}{5} q_0 \delta \cdot \tau_c(0)^{-1} \quad (46)$$

describes the time evolution of the temperature (see below), with

$$q_0 = 2^{1/5} \Gamma(21/10) C_1 / 8 = 5^{-2/5} \sqrt{\pi} \Gamma(3/5) / 8 = 0.173318 \dots, \quad (47)$$

$$r_0 \equiv \frac{2}{3\sqrt{2}\pi} \tau_c(0)^{-1}, \quad (48)$$

$$q_1 \equiv 2^{1/10} (\omega_1 / \omega_0) = 1.53445 \dots \quad (49)$$

To obtain these equations we use the expressions for  $\mu_2(t)$ ,  $B(t)$  and for coefficients  $\mathcal{A}_n$ . Note that the characteristic time  $\tau_0$  is  $\delta^{-1} \gg 1$  times larger than the mean collision time  $\tau_c(0)$ .

We will find the solution to these equations as expansions in terms of the small dissipative parameter  $\delta$  [see Eq. (24)]:

$$u = u_0 + \delta \cdot u_1 + \delta^2 \cdot u_2 + \dots, \quad (50)$$

$$a_2 = a_{20} + \delta \cdot a_{21} + \delta^2 \cdot a_{22} + \dots \quad (51)$$

Substituting Eqs. (50), (51), (38), and (39) into Eqs. (44) and (45), one can solve these equations perturbatively, for each order of  $\delta$ . Collecting terms of the order of  $O(1)$ , we obtain

$$\dot{u}_0 + \tau_0^{-1} \left( \frac{5}{3} + \frac{2}{5} a_{20} + \frac{7}{500} a_{20}^2 \right) u_0^{8/5} = 0, \quad (52)$$

$$\dot{a}_{20} + r_1 u_0^{1/2} \left( a_{20} + \frac{1}{32} a_{20}^2 \right) = 0, \quad (53)$$

where

$$r_1 \equiv \frac{1}{5} r_0 \mathcal{B}_2 = \frac{8}{15} \tau_c(0)^{-1}, \quad (54)$$

and we use the definition of  $r_0$ , and expressions (43) for  $\mathcal{B}_2$  and  $\mathcal{B}_3$ , which are zero-order coefficients in the expansion of  $\mu_4$  on  $\delta$ . Changing variables

$$t \rightarrow \tau = r_1 \int_0^t dt' u_0^{1/2}(t') \quad (55)$$

in Eq. (53), one finds the solution of this (Riccati) equation:

$$a_{20}(t) = \frac{a_{20}(0)}{\left[1 + \frac{1}{32}a_{20}(0)\right]e^{\tau} - \frac{1}{32}a_{20}(0)}. \quad (56)$$

According to Eq. (52) the characteristic time scale for  $u_0(t)$  is  $\tau_0 \gg \tau_c(0)$ , therefore; for  $t \sim \tau_c(0) \ll \tau_0$  one can approximate  $u(t) = T(t)/T_0 \approx 1$ . Moreover, if the initial deviation from the Maxwellian distribution is not large, i.e.,  $a_{20}(0)/32 \ll 1$ , for this time interval one can approximate

$$a_{20}(t) \approx a_{20}(0)e^{-4t/5\tau_E(0)}, \quad (57)$$

with  $\tau_E = \frac{3}{2}\tau_c$  being the Enskog relaxation time. Therefore,  $a_{20}(t)$  vanishes for  $t \sim \tau_0 \gg \tau_c(0)$ . This refers to the relaxation of an initially non-Maxwellian velocity distribution to the Maxwellian one. Note that the relaxation occurs within few collisions per particle, similarly to the relaxation of molecular gases.

We now assume that the initial distribution is Maxwellian, i.e., that  $a_{20}(0) = 0$  for  $t = 0$ . Then the deviation from the Maxwellian distribution originates from the inelasticity of the particle collisions. For the case  $a_{20}(0) = 0$  [and thus  $a_2(t) = 0$ ; see Eq. (56)] the solution to Eq. (52) reads

$$u_0(t) = \frac{T(t)}{T_0} = \left(1 + \frac{t}{\tau_0}\right)^{-5/3}, \quad (58)$$

which coincides with the time dependence of the temperature obtained previously using scaling arguments [20] (up to a constant  $\tau_0$  which may not be determined by scaling arguments).

For the order  $O(\delta)$ , we obtain

$$\dot{u}_1 + \frac{8}{3\tau_0}u_0^{3/5}u_1 + \frac{2}{5\tau_0}u_0^{8/5}a_{21} - \frac{5}{3\tau_0}q_1u_0^{17/10} = 0, \quad (59)$$

$$\dot{a}_{21} + r_1u_0^{1/2}a_{21} + r_2u_0^{3/5} = 0, \quad (60)$$

with

$$r_2 \equiv \left(\frac{4}{15}\right)2^{1/10}(8\pi)^{-1/2}(\mathcal{B}_4 - 5\mathcal{A}_1)\tau_c^{-1}(0). \quad (61)$$

For  $t \ll \tau_0$  we have  $u_0 \approx 1$ , and Eq. (60) reduces to

$$\dot{a}_{21} + r_1a_{21} = -r_2, \quad (62)$$

with the solution

$$a_{21}(t) = -\frac{r_2}{r_1}(1 - e^{-r_1t}) = -h(1 - e^{-4t/5\tau_E(0)}), \quad (63)$$

where

$$h \equiv r_2/r_1 = \left(\frac{3}{10}\right)\Gamma\left(\frac{21}{10}\right)2^{1/5}C_1 = 0.415964, \quad (64)$$

and we used the definitions of  $r_1$  and  $r_2$  and the values of  $\mathcal{A}_k$  and  $\mathcal{B}_k$  given above. As follows from Eq. (63), after a transient time of the order of few collisions per particle, i.e., for

$\tau_E(0) < t \ll \tau_0$ ,  $a_2(t)$  saturates to the value  $a_2 = -h\delta = -0.415964\delta$ , i.e., it changes only slowly on the time scale  $\sim \tau_c(0)$ .

For  $t \gg \tau_0$  the rescaled temperature varies as  $u_0 \approx (t/\tau_0)^{-5/3}$  [see Eq. (58)], and Eq. (60) reads

$$\dot{a}_{21} + r_1(t/\tau_0)^{-5/6}a_{21} = -r_2(t/\tau_0)^{-1}. \quad (65)$$

Using the power-law ansatz

$$a_{21}(t) \sim (t/\tau_0)^{-\nu}, \quad (66)$$

the asymptotic analysis of Eq. (65) yields the exponent  $\nu = 1/6$  and an estimate for the prefactor. Thus we find, for  $t \gg \tau_0$ ,

$$a_{21}(t) = -\frac{r_2}{r_1}(t/\tau_0)^{-1/6} = -h(t/\tau_0)^{-1/6}. \quad (67)$$

Therefore,  $a_{21}(t)$  decays to zero on the time scale  $\sim \tau_0$ , i.e., slowly on the time scale  $\sim \tau_c(0) \ll \tau_0$ . The velocity distribution thus tends asymptotically to the Maxwellian distribution.

One can also find the general solution of Eq. (60):

$$a_{21}(t) = -6\tau_0r_2 \exp\{-6\tau_0r_1(1+t/\tau_0)^{1/6}\} \times \int_{6\tau_0r_1}^{6\tau_0r_1(1+t/\tau_0)^{1/6}} \frac{e^x}{x} dx. \quad (68)$$

Noting that

$$6\tau_0r_1 = (q_0\delta)^{-1}, \quad (69)$$

$$6\tau_0r_2 = \frac{12}{5}\delta^{-1} \quad (70)$$

due to the definitions of  $r_1$ ,  $r_2$ , and  $\tau_0$ , for  $a_2(t) = \delta a_{21}(t)$  one can write in an approximation linear with respect to  $\delta$ ,

$$a_2(t) = -\frac{12}{5}w(t)^{-1}\{\text{Li}[w(t)] - \text{Li}[w(0)]\}, \quad (71)$$

where

$$w(t) \equiv \exp[(q_0\delta)^{-1}(1+t/\tau_0)^{1/6}], \quad (72)$$

and  $\text{Li}(x)$  is the logarithmic integral. It is not difficult to show that from the general expression (71) both limiting dependencies (63) for  $t \ll \tau_0$  and (67) for  $t \gg \tau_0$  are reproduced.

We could not find the general solution for  $u_1(t)$ , however, one can obtain the solution for  $t \gg \tau_0$ . Substituting asymptotic expressions  $u_0(t) \approx (t/\tau_0)^{-5/3}$  and  $a_{21}(t) \approx -h(t/\tau_0)^{-1/6}$  into Eq. (59) for  $u_1(t)$ , we recast this equation into the form

$$\dot{u}_1 + \frac{8}{3}(t/\tau_0)^{-1}u_1 = \left(\frac{2}{5}h + \frac{5}{3}q_1\right)(t/\tau_0)^{-17/6}. \quad (73)$$

Again a power-law ansatz  $u_1(t) \sim (t/\tau_0)^\alpha$  allows us to obtain both the exponent  $\alpha = 11/6$  as well as the corresponding prefactor. The result for  $u_1(t)$  for  $t \gg \tau_0$  reads

$$u_1(t) = \left( \frac{12}{25}h + 2q_1 \right) (t/\tau_0)^{-11/6} = 3.26856(t/\tau_0)^{-11/6}, \quad (74)$$

where we used the above results for the constants  $h$  and  $q_1$ . From the last equation one can see how the coupling between the temperature and the velocity distribution influences the evolution of the temperature. Indeed, if there were no such coupling, there would be no coupling term in Eq. (59), and thus, no contribution from  $\frac{12}{25}h$  to the prefactor of  $u_1(t)$  in Eq. (74). This would noticeably change the time behavior of  $u_1(t)$ . On the other hand, the leading term in the time dependence of temperature,  $u_0(t)$ , is not affected by this kind of coupling.

In Figs. 1 and 2 we show the time dependence of the coefficient  $a_2(t)$  of the Sonine polynomial expansion and of the temperature of the granular gas. The analytical findings are compared with the numerical solution of the system of equations (44) and (45). As follows from the figures, the analytical theory reproduces fairly well the numerical results for the case of small  $\delta$ .

As follows from Fig. 1, for small  $\delta$  the following scenario of evolution of the velocity distribution takes place for a force-free granular gas. The initial Maxwellian distribution evolves to a non-Maxwellian distribution, with the discrepancy between these two characterized by the second coefficient of the Sonine polynomials expansion  $a_2$ . The deviation from the Maxwellian distribution (described by  $a_2$ ) quickly grows, until it saturates after a few collisions per particle at a “steady-state” value. At this instant the deviation from the Maxwellian distribution is maximal, with the value  $a_2 \approx -0.4\delta$  (Fig. 1, top). This refers to the first “fast” stage of the evolution, which takes place on a mean-collision time scale  $\sim \tau_c(0)$ . After this maximal deviation is reached, the second “slow” stage of the evolution starts. At this stage  $a_2$  decays to zero on a “slow” time scale  $\tau_0 \sim \delta^{-1}\tau_c(0) \gg \tau_c(0)$ , which corresponds to the time scale of the temperature evolution (Fig. 1, middle); the decay of the coefficient  $a_2(t)$  in this regime occurs according to a power law  $\sim t^{-1/6}$  (Fig. 1, bottom). Asymptotically the Maxwellian distribution would be achieved, if the clustering process did not occur.

Figure 2 illustrates the significance of the first-order correction  $u_1(t)$  in the time evolution of the temperature. This becomes more important as the dissipation parameter  $\delta$  grows (Fig. 2, top and middle). At large times the results of the first-order theory [with  $u_1(t)$  included] practically coincide with the numerical results, while zero-order theory [without  $u_1(t)$ ] demonstrates noticeable deviations (Fig. 2, bottom).

According to our analysis of the diffusion in granular gas of viscoelastic particles [19], the clustering is expected to be retarded, compared to the case of a constant  $\epsilon$ . Therefore, we may assume that for the time shown in the figures the granular gas is still in the regime of homogeneous cooling.

For larger values of  $\delta$  the linear theory breaks down. Unfortunately, the equations obtained for the second order approximation  $O(\delta^2)$  are too complicated to be treated analytically. Hence, we studied them only numerically (see Fig. 3). As compared to the case of small  $\delta$ , an additional intermediate regime in the time evolution of the velocity distribution is observed. The first “fast” stage of evolution takes place,

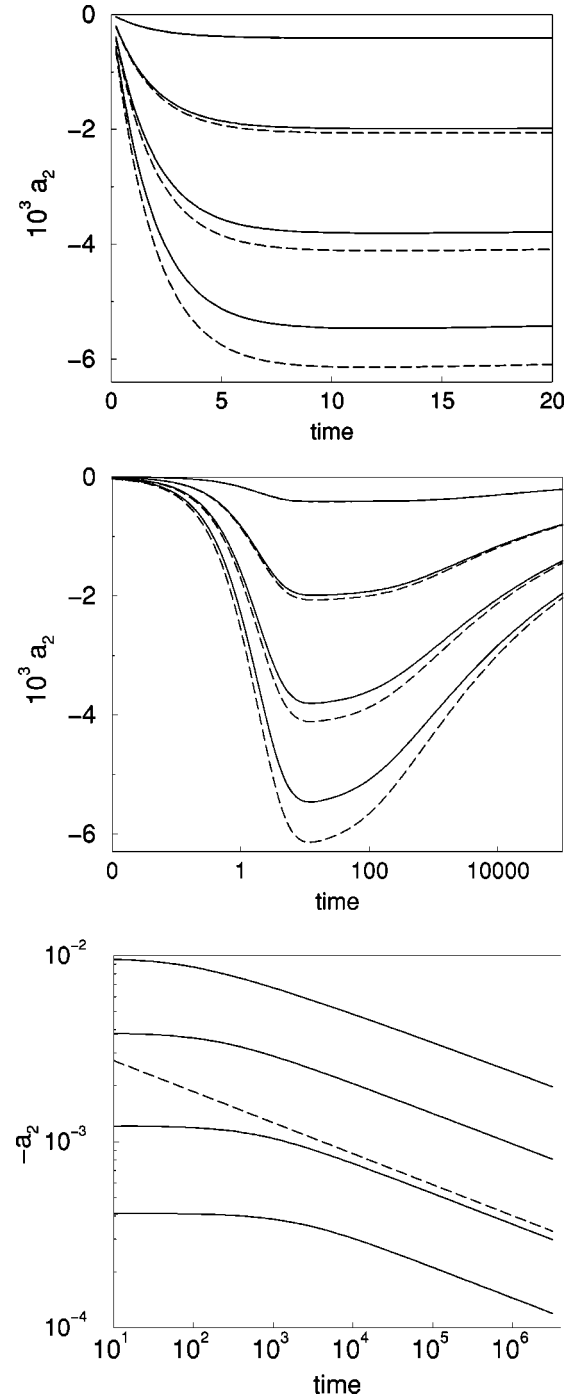


FIG. 1. Time dependence of the second coefficient of the Sonine polynomial expansion  $a_2(t)$ . Time is given in units of the mean collisional time  $\tau_c(0)$ . Top:  $a_2 \times 1000$  (solid lines) for  $\delta = 0.001, 0.005, 0.01$ , and  $0.015$  (top to bottom), together with the linear approximation (dashed lines). Middle: the same as the top but for larger times. Bottom:  $-a_2(t)$  over time (log scale) for  $\delta = 0.03, 0.01, 0.003$ , and  $0.001$  (top to bottom), together with the power-law asymptotics  $\sim t^{-1/6}$ .

as before, on a time scale of few collisions per particle, where maximal deviation from the Maxwellian distribution is achieved (Fig. 3). For  $\delta \geq 0.15$  these maximal values of  $a_2$  are positive. Then, in the second stage (intermediate regime), which continues 10–100 collisions,  $a_2$  changes its sign and reaches a maximal negative deviation. Finally, on the third,

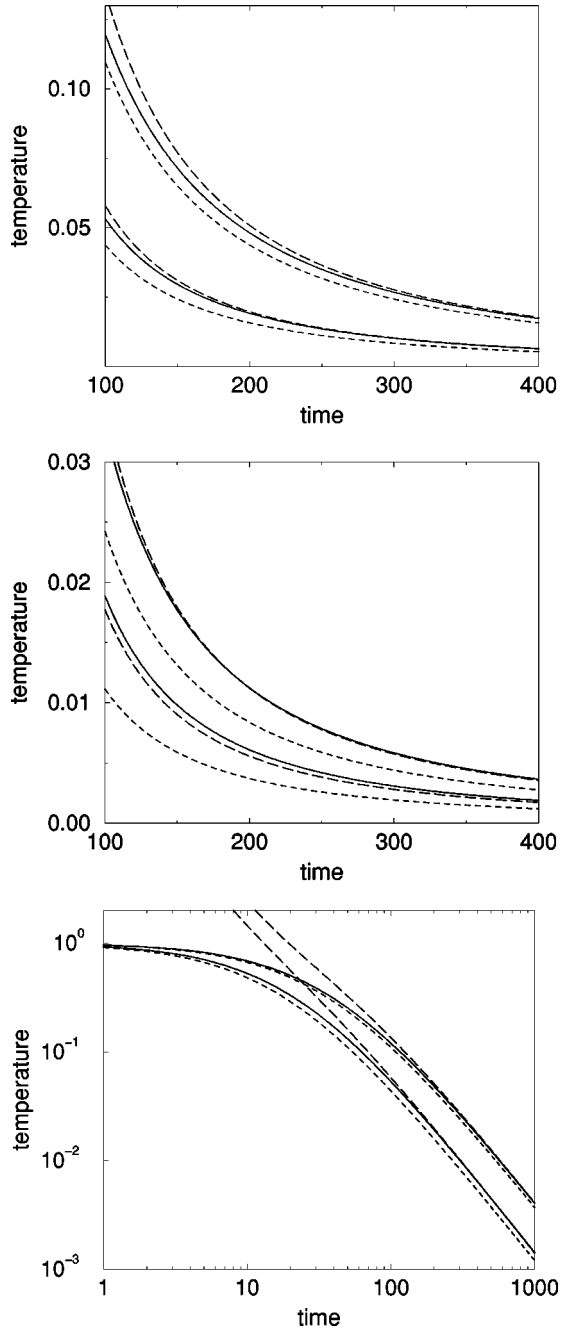


FIG. 2. Time evolution of the reduced temperature,  $u(t) = T(t)/T_0$ . The time is given in units of the mean collisional time  $\tau_c(0)$ . Solid line: numerical solution, short-dashed line:  $u_0(t) = (1 + t/\tau_0)^{-5/3}$  (zero-order theory), long-dashed line:  $u(t) = u_0(t) + \delta u_1(t)$  (first-order theory). Top  $\delta = 0.05$  and  $0.1$  (top to bottom). Middle:  $\delta = 0.15$  and  $0.25$  (top to bottom). Bottom: the same as the top but for log scale and larger ranges.

slow stage,  $a_2(t)$  relaxes to zero on a slow time scale  $\sim \tau_0$ , just as for small  $\delta$ . In Fig. 3 we show the first stage of the time evolution of  $a_2(t)$  for systems with large  $\delta$ . At a certain value of the dissipative parameter  $\delta$  the behavior changes qualitatively, i.e. the system then reveals another time scale, as discussed above.

Figure 4 shows the numerical solution of Eqs. (44) and (45) for the second Sonine coefficient  $a_2(t)$  as a function of time. One can clearly distinguish the different stages of evolution of the velocity distribution function.

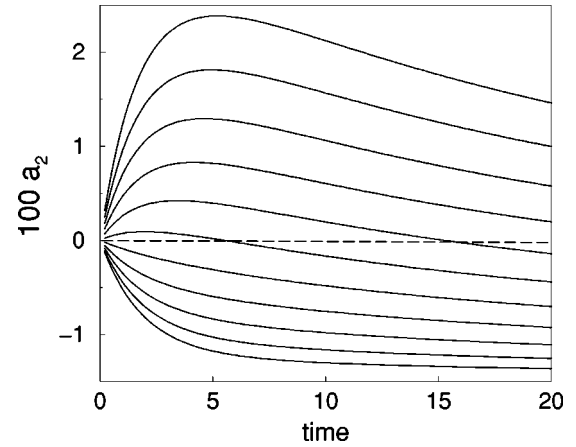


FIG. 3. Time dependence of the second coefficient of the Sonine polynomial expansion  $a_2(t) \times 100$ . Time is given in units of mean collisional time  $\tau_c(0)$ .  $\delta = 0.1, 0.11, 0.12, \dots, 0.20$  (bottom to top).

Thus we conclude that for the case of a not very small dissipative parameter  $\delta$ , the time evolution of the velocity distribution function (described on the level of the second coefficient of the Sonine polynomials expansion) exhibits a very complicated nonmonotonic behavior with a few different regimes. Physically such behavior is caused by the existence of an additional intrinsic time scale which describes the viscoelastic collision, and by the coupling of the evolution of the velocity distribution with the time evolution of the temperature.

The analysis performed up to now has addressed the main part of the velocity distribution function. The most important component of the distribution is still the Maxwellian, while deviations from this have been quantified in terms of the Sonine polynomial expansion. For very large velocities, however, this is not true and the Maxwellian distribution may not be used as a zero-order approximation. In Sec. V we address the problem of properties of the velocity distribution function for  $v \gg v_0$ .

## V. HIGH-VELOCITY TAIL OF THE VELOCITY-DISTRIBUTION FUNCTION

The high-velocity tail of the velocity distribution function in force-free granular gases was analyzed for the case of a constant restitution coefficient in Refs. [5,6]. It was shown in these studies that for large velocities,  $c \gg 1$ , the velocity distribution function behaves as  $\tilde{f}(c) \sim \exp(-\text{const } c)$ , i.e., that the tail  $c \gg 1$  is overpopulated, as compared to the Maxwellian distribution  $\sim \exp(-c^2)$ .

Here we use the scheme of analysis proposed in Ref. [5]. The same arguments as in Refs. [5,6], lead to a conclusion that the gain term of the collisional integral  $\tilde{I}$  may be neglected for  $c \gg 1$  with respect to the loss term, which does not depend on the restitution coefficient. Thus, following Refs. [5,6], we approximate the collision integral as

$$\tilde{I}(\tilde{f}, \tilde{f}) \approx -\pi c \tilde{f}(\vec{c}, t), \quad (75)$$

and for  $c \gg 1$  write the kinetic equation (27) as



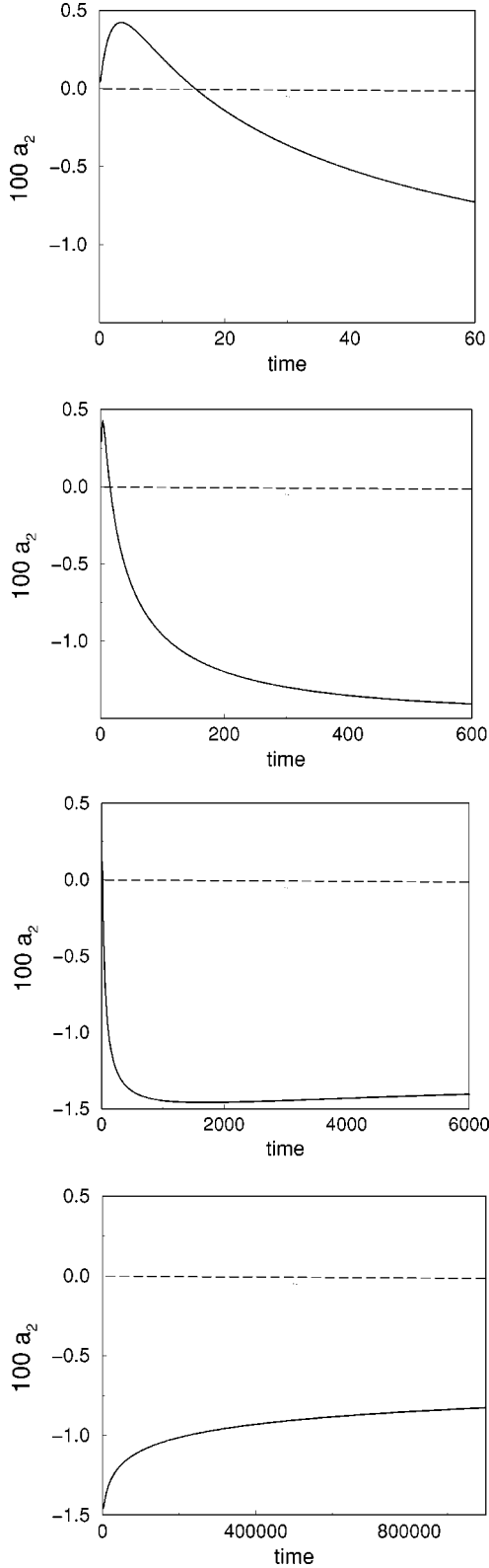


FIG. 4. The second Sonine coefficient  $a_2$  for  $\delta=0.16$  over time. The numerical solutions of Eqs. (44) and (45) show all stages of evolution discussed in the text.

$$\frac{\mu_2}{3} c \frac{\partial}{\partial c} \tilde{f}(\vec{c}, t) + B^{-1} \frac{\partial}{\partial t} \tilde{f}(\vec{c}, t) \approx -\pi c \tilde{f}(\vec{c}, t). \quad (76)$$

If one would use expansion (20) (with coefficients  $a_p$  for

$p > 2$  discarded) to substitute it into Eq. (76) for the *second* term on the left-hand side of (76) at  $c \gg 1$  one would obtain

$$B^{-1} \frac{\partial \tilde{f}}{\partial t} = \frac{4}{3} \left[ \mu_2(1+a_2) - \frac{1}{5} \mu_4 \right] \phi(c) S_2(c^2) \sim c^4 e^{-c^2}, \quad (77)$$

where we have used the relation

$$\partial \tilde{f} / \partial t = \dot{a}_2 \phi(c) S_2(c^2), \quad (78)$$

with  $\dot{a}_2$  according to Eq. (34), and definition (11) of  $S_2(c^2)$ , which shows that  $S_2(c^2) \sim c^4$  at  $c \gg 1$ . We also take into account that  $\mu_2$ ,  $\mu_4$ , and  $a_2$  do not depend on  $c$ . For the first term in the left-hand side of Eq. (76) and for the right-hand side of Eq. (76), this substitute yields, correspondingly in the same limit  $c \gg 1$ ,

$$c \frac{\partial \tilde{f}}{\partial c} \sim -c^6 e^{-c^2}, \quad (79)$$

$$c \tilde{f} \sim c^5 e^{-c^2}. \quad (80)$$

From Eqs. (77), (79), and (80) it follows that, although all terms in the Eq. (76) have the same factor  $e^{-c^2}$ , the exponents of the power of  $c$  of the prefactor are different for all terms. This means an inconsistency of substitution (20), with  $a_p$  for  $p > 2$  discarded, for  $c \gg 1$ . Similarly, it may be shown that such an inconsistency appears for any order of the Sonine polynomial expansion. Indeed, using the Sonine polynomial expansion (20) up to (arbitrary) order  $n$  yields the estimate  $\sim c^{(2n+2)} e^{-c^2}$  for the first term and  $\sim c^{2n} e^{-c^2}$  for the second term on the left-hand side of Eq. (76), while for the right-hand side of Eq. (76) one obtains  $\sim c^{(2n+1)} e^{-c^2}$ .

However, the exponential ansatz

$$\tilde{f}(\vec{c}, t) \sim \exp\{-\varphi(t) \cdot c\} \quad (81)$$

for the kinetic equation (76) turns out to be self-consistent for  $c \gg 1$ . Substituting this into Eq. (76), one finds that the function  $\varphi(t)$  in Eq. (81) must satisfy

$$\dot{\varphi} + \frac{1}{3} \mu_2 B \varphi = \pi B, \quad (82)$$

where the time dependence of  $B$  is given by Eq. (35), and  $\mu_2$  depends on time via  $a_2(t)$  according to Eq. (38). In the approximation  $a_2 \sim \delta$ , linear with respect to  $\delta$ , and therefore, according to Eqs. (38) and (40),  $\mu_2(t) \approx \delta' \omega_0$ . Then, using the definition (24) of  $\delta'$  and expression (35) for  $B(t)$ , one obtains

$$\mu_2(t) B(t) = \frac{5}{2} \tau_0^{-1} u^{3/5}(t), \quad (83)$$

$$B(t) = \frac{\tau_c(0)^{-1}}{\sqrt{8\pi}} u^{1/2}(t),$$

with  $\tau_0$  being defined by Eq. (46). With Eq. (58) for the time dependence of the temperature in this approximation, Eq. (82) reads

$$\dot{\varphi} + \frac{5}{6\tau_0} \left(1 + \frac{t}{\tau_0}\right)^{-1} \varphi = \sqrt{\frac{\pi}{8}} \tau_c(0)^{-1} \left(1 + \frac{t}{\tau_0}\right)^{-5/6}. \quad (84)$$

Substituting the ansatz  $\varphi \sim (1 + t/\tau_0)^\nu$ , we find the exponent  $\nu = 1/6$  and the prefactor, so we arrive at the final result

$$\varphi(t) = b \delta^{-1} \left(1 + \frac{t}{\tau_0}\right)^{1/6}, \quad (85)$$

with

$$b = \sqrt{\frac{\pi}{2}} \left(\frac{5}{16q_0}\right) = \frac{5^{7/5}}{2^{3/2}\Gamma(3/5)} = 2.25978 \dots \quad (86)$$

Thus the velocity distribution function reads, for  $c \gg 1$ ,

$$\tilde{f}(\vec{c}, t) \sim \exp\left[-\frac{b}{\delta} c \left(1 + \frac{t}{\tau_0}\right)^{1/6}\right]. \quad (87)$$

Note that the obtained expression (87) refers only for times  $t \gg \tau_c(0)$ , when the deviations from the Maxwellian distribution are already well developed; it is not applicable for the transient times  $t \sim \tau_c(0)$ .

As one can see from Eq. (87) the overpopulation (with respect to the Maxwellian distribution) of the high-velocity tail decreases with time on the same time scale  $\sim \tau_0$  as  $a_2(t)$ , i.e., the velocity distribution in the system approaches the Maxwellian. However, it should be noted that the above considerations are valid as long as the overpopulation in the tail is significant enough to make the gain term in the collision integral negligible as compared to the loss term.

## VI. CONCLUSION

We studied the velocity distribution in a homogeneously cooling granular gas of viscoelastic particles, which implies an impact-velocity-dependent restitution coefficient. We observed that, contrary to the case of the constant restitution coefficient, the distribution function may not be represented in a simple scaling form, where the time dependence of the function occurs only via the time dependence of the temperature. The dependence of the restitution coefficient on the impact velocity causes a time dependence of the coefficients of the Sonine polynomial expansion, which describes the deviation of the velocity distribution from the Maxwellian.

We analyzed the time evolution of the temperature and of the second coefficient of the Sonine polynomials expansion  $a_2$ . Contrary to the case of the constant restitution coefficient, the evolution of temperature is now coupled to the time evolution of  $a_2$ .

For small values of the dissipative parameter  $\delta$ , we developed an analytical theory for the time evolution of the temperature of the granular gas and for the coefficient of the Sonine polynomial expansion  $a_2$ ; the case of larger  $\delta$  was studied numerically. We observed a complicated nonmonotonic time behavior of the coefficient  $a_2$ . For small values of

the dissipative parameter  $\delta$  we detected two different stages in its time evolution: a first fast stage, which develops on the time scale of the mean-collision time  $\tau_c$ ; and a second, slow stage on the time scale  $\sim \tau_0 \gg \tau_c$ , on which the temperature of the granular gas changes. In the fast stage a maximal deviation from the Maxwellian distribution is achieved, and then the deviation relaxes to zero during the second slow stage. Our numerical results agree well with the predictions of the analytical theory for small  $\delta$ .

When  $\delta$  is not small, a much more complicated time behavior of the coefficient  $a_2$  has been revealed. In addition to the two stages of evolution which were observed for the case of a small dissipative parameter, a regime of intermediate relaxation has been detected. Physically such complicated behavior is caused by an additional intrinsic time scale which describes the viscoelastic collision, and by a coupling of the evolution of the velocity distribution to the time evolution of temperature.

We also analyzed the high-velocity tail of the velocity distribution for the case of the impact-velocity-dependent restitution coefficient for viscoelastic particles. We found the same exponential overpopulation for the tail as for the constant restitution coefficient. However, contrary to the latter case, where the overpopulation of the tail persists with time, it decreases for the impact dependent restitution coefficient, and the velocity distribution tends to become Maxwellian as the system evolves.

The homogeneous cooling state is the simplest part of the evolution of a granular gas which precedes all other evolution steps, which involve spontaneous cluster formation, complicated structure formation in the velocity field, etc. Presently we are far from being able to develop a full theory of the evolution of granular gases, beginning from their generation and extending to a fully developed variety of structures. Therefore, we consider the results for the homogeneous cooling state only as the first step toward understanding the complicated evolution of granular gases. Astrophysical systems such as planetary rings (see, e.g., Ref. [25]) possibly originate from catastrophic impacts of heavy objects with a planet, which generates a cloud of dust. This cloud will then undergo an evolution whose first regime might be considered a homogeneously cooling gas. As follows from results reported here, the evolution of the velocity distribution and temperature even in this simplest state is far from being trivial.

## ACKNOWLEDGMENTS

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## APPENDIX A: DERIVATION OF EQ. (17)

The change of the particle velocities due to the inverse collision is described by

$$\vec{v}_1 = \vec{v}_1^{**} - \frac{1}{2}[1 + \epsilon(g^{**})]g^{**}\vec{e}, \quad (\text{A1})$$

$$\vec{v}_2 = \vec{v}_2^{**} + \frac{1}{2}[1 + \epsilon(g^{**})]g^{**}\vec{e},$$

where we introduce the normal relative velocity  $g^{**} \equiv \vec{v}_{12}^{**} \cdot \vec{e}$ , and where

$$\epsilon(g^{**}) = 1 - C_1 A \alpha^{2/5} |g^{**}|^{1/5} + C_2 A^2 \alpha^{4/5} |g^{**}|^{2/5} \mp \dots, \quad (\text{A2})$$

according to the viscoelastic character of the particles [see Eq. (2)]. Equations (A1) and (A2) imply the conservation of momentum

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_1^{**} + \vec{v}_2^{**}, \quad (\text{A3})$$

and the relation

$$g = -\epsilon(g^{**})g^{**}, \quad (\text{A4})$$

with  $g \equiv \vec{v}_{12} \cdot \vec{e}$ . Using  $C_2 = \frac{3}{5} C_1^2$  [Eq. (5)], one can also write

$$g^{**} = -g \left[ 1 + C_1 A \alpha^{2/5} |g|^{1/5} + \frac{3}{5} (C_1 A \alpha^{2/5})^2 |g|^{2/5} \mp \dots \right]. \quad (\text{A5})$$

We use Eq. (A5) to find the relation between the length of the collisional cylinders,  $|g|dt$  and  $|g^{**}|dt$ , when the transformation of variables  $\vec{v}_1^{**}, \vec{v}_2^{**} \rightarrow \vec{v}_1, \vec{v}_2$  is made in the collisional integral. One also needs the Jacobian of this transformation. To calculate this, it is convenient to choose the coordinate axis  $Z$  along the intercenter vector  $\vec{e}$ , i.e.,

$$g \equiv v_{12,z} \equiv v_{1,z} - v_{2,z}. \quad (\text{A6})$$

Then, from Eqs. (A1), it follows that

$$\begin{aligned} v_{1,x}^{**} &= v_{1,x}, & v_{1,y}^{**} &= v_{1,y}, \\ v_{2,x}^{**} &= v_{2,x}, & v_{2,y}^{**} &= v_{2,y}, \\ v_{1,z}^{**} &= v_{1,z} + \frac{1}{2}(g^{**} - g), \\ v_{2,z}^{**} &= v_{2,z} - \frac{1}{2}(g^{**} - g), \end{aligned} \quad (\text{A7})$$

where the value of  $g^{**}$  is expressed in terms of  $g$  (i.e., in terms of  $v_{1,z}$  and  $v_{2,z}$ ) according to Eq. (A5). Thus Eqs. (A7) explicitly express all components of the inverse-collision velocities  $\vec{v}_1^{**}$  and  $\vec{v}_2^{**}$  in terms of  $\vec{v}_1$  and  $\vec{v}_2$ . For the Jacobian, straightforward calculations yield

$$\begin{aligned} d\vec{v}_1^{**} d\vec{v}_2^{**} &= \left[ 1 + \frac{6}{5} C_1 A \alpha^{2/5} |g|^{1/5} \right. \\ &\quad \left. + \frac{21}{25} (C_1 A \alpha^{2/5})^2 |g|^{2/5} \mp \dots \right] d\vec{v}_1 d\vec{v}_2. \end{aligned} \quad (\text{A8})$$

Combining Eq. (A8) with Eq. (A5), which relates the lengths of collisional cylinders, one arrives at the factor  $\chi$  [Eq. (17)] in the collisional integral.

## APPENDIX B: DERIVATION OF THE MOMENTS $\mu_p$ [EQS. (38) AND (39)]

To calculate the moments

$$\begin{aligned} \mu_p &= -\frac{1}{2} \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e} \Theta(-\vec{c}_{12} \cdot \vec{e}) |\vec{c}_{12} \cdot \vec{e}| \phi(c_1) \phi(c_2) \\ &\quad \times \{1 + a_2 [S_2(c_1^2) + S_2(c_2^2)] \\ &\quad + a_2^2 S_2(c_1^2) S_2(c_2^2)\} \Delta(c_1^p + c_2^p), \end{aligned} \quad (\text{B1})$$

it is convenient to use the center of mass velocity  $\vec{C}$  and relative velocity  $\vec{c}_{12}$ , such that

$$\vec{c}_1 = \vec{C} + \frac{1}{2}\vec{c}_{12}, \quad \vec{c}_2 = \vec{C} - \frac{1}{2}\vec{c}_{12}. \quad (\text{B2})$$

The Jacobian of transformation (B2) is equal to unity, and the product  $\phi(\vec{c}_1)\phi(\vec{c}_2)$  transforms into

$$\begin{aligned} \phi(\vec{c}_1)\phi(\vec{c}_2) &\rightarrow \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2}c_{12}^2\right) \left(\frac{2}{\pi}\right)^{3/2} \exp(-2C^2) \\ &\equiv \phi(\vec{c}_{12})\phi(\vec{C}). \end{aligned} \quad (\text{B3})$$

In terms of the variables  $\vec{C}$  and  $\vec{c}_{12}$  the quantity  $[S_2(c_1^2) + S_2(c_2^2)]$  in Eq. (B1) may be written as

$$\begin{aligned} [S_2(c_1^2) + S_2(c_2^2)] &= C^4 + (\vec{C} \cdot \vec{c}_{12})^2 + \frac{1}{16}c_{12}^4 + \frac{1}{2}C^2c_{12}^2 - 5C^2 \\ &\quad - \frac{5}{4}c_{12}^2 + \frac{15}{4}. \end{aligned} \quad (\text{B4})$$

For  $S_2(c_1^2)S_2(c_2^2)$ , we obtain

$$S_2(c_1^2)S_2(c_2^2) = K_1 + K_2 + K_3 + K_4, \quad (\text{B5})$$

where

$$K_1 = \frac{1}{4}C^8 - \frac{5}{2}C^6 + \frac{65}{8}C^4 - \frac{75}{8}C^2, \quad (\text{B6})$$

$$K_2 = \frac{1}{1024}c_{12}^8 - \frac{5}{128}c_{12}^6 + \frac{65}{128}c_{12}^4 - \frac{75}{32}c_{12}^2, \quad (\text{B7})$$

$$\begin{aligned} K_3 &= \frac{3}{32}C^4c_{12}^4 + \frac{1}{4}C^6c_{12}^2 + \frac{1}{64}C^2c_{12}^6 - \frac{15}{8}C^4c_{12}^2 \\ &\quad - \frac{15}{32}C^2c_{12}^4 + \frac{65}{16}C^2c_{12}^2, \end{aligned} \quad (\text{B8})$$

$$\begin{aligned}
K_4 &= \frac{1}{4}(\vec{C} \cdot \vec{c}_{12})^4 - \frac{1}{4}C^2(\vec{C} \cdot \vec{c}_{12})^2 c_{12}^2 - \frac{1}{2}C^4(\vec{C} \cdot \vec{c}_{12})^2 \\
&\quad - \frac{1}{32}(\vec{C} \cdot \vec{c}_{12})^2 c_{12}^4 + \frac{5}{2}C^2(\vec{C} \cdot \vec{c}_{12})^2 + \frac{5}{8}(\vec{C} \cdot \vec{c}_{12})^2 c_{12}^2 \\
&\quad - \frac{35}{8}(\vec{C} \cdot \vec{c}_{12})^2 + \left(\frac{15}{8}\right)^2. \tag{B9}
\end{aligned}$$

For the quantities  $\Delta(c_1^p + c_2^p)$  ( $p=2$  and  $4$ ), we find

$$\Delta(c_1^2 + c_2^2) = -\frac{1}{2}(\vec{c}_{12} \cdot \vec{e})^2(1 - \epsilon^2) \tag{B10}$$

and

$$\begin{aligned}
\Delta(c_1^4 + c_2^4) &= 2(1 + \epsilon)^2(\vec{c}_{12} \cdot \vec{e})^2(\vec{C} \cdot \vec{e})^2 + \frac{1}{8}(1 - \epsilon^2)^2(\vec{c}_{12} \cdot \vec{e})^4 \\
&\quad - \frac{1}{4}(1 - \epsilon^2)(\vec{c}_{12} \cdot \vec{e})^2 \vec{c}_{12}^2 - \vec{C}^2(1 - \epsilon^2)(\vec{c}_{12} \cdot \vec{e})^2 \\
&\quad - 4(1 + \epsilon)(\vec{C} \cdot \vec{c}_{12})(\vec{C} \cdot \vec{e})(\vec{c}_{12} \cdot \vec{e}). \tag{B11}
\end{aligned}$$

Substituting Eqs. (B4), (B5), (B10), and (B11) into Eq. (B1), and using the expansions

$$(1 - \epsilon^2) = 2C_1 \delta'(t) |\vec{c}_{12} \cdot \vec{e}|^{1/5} - \frac{11}{5}C_1^2 \delta'^2(t) |\vec{c}_{12} \cdot \vec{e}|^{2/5} + \dots, \tag{B12}$$

$$\begin{aligned}
(1 + \epsilon)^2 &= 4 \left[ 1 - C_1 \delta'(t) |\vec{c}_{12} \cdot \vec{e}|^{1/5} \right. \\
&\quad \left. + \frac{17}{20}C_1^2 \delta'^2(t) |\vec{c}_{12} \cdot \vec{e}|^{2/5} + \dots \right], \tag{B13}
\end{aligned}$$

$$(1 - \epsilon^2)^2 = 4C_1^2 \delta'^2(t) |\vec{c}_{12} \cdot \vec{e}|^{2/5} + \dots, \tag{B14}$$

one observes that  $\mu_2$  and  $\mu_4$  may be expressed in terms of the basic integrals

$$\begin{aligned}
J_{k,l,m,n,p,\alpha} &= \int d\vec{c}_{12} \int d\vec{C} \int d\vec{e} \Theta(-\vec{c}_{12} \cdot \vec{e}) |\vec{c}_{12} \cdot \vec{e}|^{1+\alpha} \\
&\quad \times \phi(c_{12}) \phi(C) C^k c_{12}^l (\vec{C} \cdot \vec{c}_{12})^m (\vec{C} \cdot \vec{e})^n (\vec{c}_{12} \cdot \vec{e})^p. \tag{B15}
\end{aligned}$$

That is, for  $\mu_2$  one has

$$\begin{aligned}
\mu_2 &= \frac{1}{2} \delta' C_1 \left[ J_{0,0,0,0,2,1/5} + a_2 L \left( \frac{1}{5} \right) + a_2^2 M \left( \frac{1}{5} \right) \right] \\
&\quad - \frac{11}{20} \delta'^2 C_1^2 \left[ J_{0,0,0,0,2,2/5} + a_2 L \left( \frac{2}{5} \right) + a_2^2 M \left( \frac{2}{5} \right) \right] \tag{B16}
\end{aligned}$$

where we define

$$\begin{aligned}
L(\alpha) &= J_{4,0,0,0,2,\alpha} + J_{0,0,2,0,2,\alpha} + \frac{1}{16} J_{0,4,0,0,2,\alpha} + \frac{1}{2} J_{2,2,0,0,2,\alpha} \\
&\quad - 5J_{2,0,0,0,2,\alpha} - \frac{5}{4} J_{0,2,0,0,2,\alpha} + \frac{15}{4} J_{0,0,0,0,2,\alpha} \tag{B17}
\end{aligned}$$

and

$$\begin{aligned}
M(\alpha) &= \frac{1}{4} J_{8,0,0,0,2,\alpha} - \frac{5}{2} J_{6,0,0,0,2,\alpha} + \frac{65}{8} J_{4,0,0,0,2,\alpha} \\
&\quad - \frac{75}{8} J_{2,0,0,0,2,\alpha} + \frac{1}{1024} J_{0,8,0,0,2,\alpha} - \frac{5}{128} J_{0,6,0,0,2,\alpha} \\
&\quad + \frac{65}{128} J_{0,4,0,0,2,\alpha} - \frac{75}{32} J_{0,2,0,0,2,\alpha} + \frac{3}{32} J_{4,4,0,0,2,\alpha} \\
&\quad + \frac{1}{4} J_{6,2,0,0,2,\alpha} + \frac{1}{64} J_{2,6,0,0,2,\alpha} - \frac{15}{8} J_{4,2,0,0,2,\alpha} \\
&\quad + \frac{65}{16} J_{2,2,0,0,2,\alpha} + \frac{1}{4} J_{0,0,4,0,2,\alpha} - \frac{1}{4} J_{2,2,2,0,2,\alpha} \\
&\quad - \frac{1}{2} J_{4,0,2,0,2,\alpha} - \frac{1}{32} J_{0,4,2,0,2,\alpha} + \frac{5}{2} J_{2,0,2,0,2,\alpha} \\
&\quad + \frac{5}{8} J_{0,2,2,0,2,\alpha} - \frac{35}{8} J_{0,0,2,0,2,\alpha} + \left(\frac{15}{8}\right)^2 J_{0,0,0,0,2,\alpha}. \tag{B18}
\end{aligned}$$

The basic integrals may be calculated (details are given in Appendix C), and the following expressions are obtained:

$$\begin{aligned}
J_{k,l,m,0,p,\alpha} &= \frac{(-1)^p \times 8 \times 2^{(-k+l+p+\alpha-1)/2}}{(p+\alpha+2)(m+1)} \\
&\quad \times [1 - (-1)^{m+1}] \Gamma\left(\frac{k+m+3}{2}\right) \\
&\quad \times \Gamma\left(\frac{l+m+p+\alpha+4}{2}\right) \tag{B19}
\end{aligned}$$

for  $n=0$ ,

$$\begin{aligned}
J_{k,l,m,1,p,\alpha} &= \frac{(-1)^{p+1} \times 4 \times 2^{(-k+l+p+\alpha)/2}}{(p+\alpha+3)(m+2)} \\
&\quad \times [1 - (-1)^m] \Gamma\left(\frac{k+m+4}{2}\right) \\
&\quad \times \Gamma\left(\frac{l+m+p+\alpha+4}{2}\right) \tag{B20}
\end{aligned}$$

for  $n=1$  and

$$J_{k,l,m,2,p,\alpha} = \frac{(-1)^p \times 4 \times 2^{(-k+l+p+\alpha-1)/2}}{(p+\alpha+2)(m+1)} \times [1 - (-1)^{m+1}] \Gamma\left(\frac{k+m+5}{2}\right) \times \Gamma\left(\frac{l+m+p+\alpha+4}{2}\right) \left[\frac{p+\alpha+1}{m+3} + \frac{1}{m+1}\right] \quad (\text{B21})$$

for  $n=2$ .

When we compare Eqs. (B16) and (38) for  $\mu_2$ , and use relations (B19), (B20), and (B21) for the basic integrals, we find

$$\mathcal{A}_1 = \frac{1}{2} C_1 J_{0,0,0,0,2,1/5} = 2\sqrt{2\pi} 2^{1/10} \Gamma\left(\frac{21}{10}\right) C_1 = 6.48562 \dots \equiv \omega_0, \quad (\text{B22})$$

$$\mathcal{A}_4 = \frac{11}{20} C_1^2 J_{0,0,0,0,2,2/5} = \sqrt{2\pi} 2^{1/5} \Gamma\left(\frac{16}{5}\right) C_1^2 = 9.28569 \dots \equiv \omega_1. \quad (\text{B23})$$

Computing from the basic integrals  $L(\alpha)$  and  $M(\alpha)$  for  $\alpha = \frac{1}{5}$  and  $\frac{2}{5}$ , and using the relation for the  $\Gamma$  function  $\Gamma(x+1) = x\Gamma(x)$ , we obtain

$$\mathcal{A}_2 = \frac{1}{2} C_1 L\left(\frac{1}{5}\right) = \frac{6}{25} \omega_0, \quad (\text{B24})$$

$$\mathcal{A}_5 = \frac{11}{20} C_1^2 L\left(\frac{2}{5}\right) = \frac{119}{400} \omega_1, \quad (\text{B25})$$

$$\mathcal{A}_3 = \frac{1}{2} C_1 M\left(\frac{1}{5}\right) = \frac{21}{2500} \omega_0, \quad (\text{B26})$$

$$\mathcal{A}_6 = \frac{11}{20} C_1^2 M\left(\frac{2}{5}\right) = \frac{4641}{640000} \omega_1. \quad (\text{B27})$$

Similar calculations may be performed for  $\mu_4$ , and yield Eq. (39), with the coefficients  $\mathcal{B}_k$  expressed in terms of the basic integrals:

$$\mathcal{B}_1 = 4(J_{0,0,1,1,1,0} - J_{0,0,0,2,2,0}) = 4(\sqrt{2\pi} - \sqrt{2\pi}) = 0, \quad (\text{B28})$$

$$\begin{aligned} \mathcal{B}_2 = & -4 \left( \frac{1}{16} J_{0,4,0,2,2,0} - J_{4,0,1,1,1,0} + J_{0,0,2,2,2,0} - J_{0,0,3,1,1,0} \right. \\ & + J_{4,0,0,2,2,0} - \frac{1}{16} J_{0,4,1,1,1,0} + \frac{1}{2} J_{2,2,0,2,2,0} - \frac{1}{2} J_{2,2,1,1,1,0} \\ & - \frac{5}{4} J_{0,2,0,2,2,0} + \frac{5}{4} J_{0,2,1,1,1,0} - 5J_{2,0,0,2,2,0} + 5J_{2,0,1,1,1,0} \\ & \left. + \frac{15}{4} J_{0,0,0,2,2,0} - \frac{15}{4} J_{0,0,1,1,1,0} \right) = 4\sqrt{2\pi}, \quad (\text{B29}) \end{aligned}$$

$$\mathcal{B}_4 = 4C_1 \left( J_{0,0,0,2,2,1/5} - \frac{1}{2} J_{0,0,1,1,1,1/5} + \frac{1}{16} J_{0,2,0,0,2,1/5} + \frac{1}{4} J_{2,0,0,0,2,1/5} \right) = \frac{56}{5} \sqrt{2\pi} 2^{1/10} \Gamma\left(\frac{21}{10}\right) C_1 = \frac{28}{5} \omega_0, \quad (\text{B30})$$

$$\begin{aligned} \mathcal{B}_7 = & C_1^2 \left( \frac{11}{10} J_{2,0,0,0,2,2/5} + \frac{11}{40} J_{0,2,0,0,2,2/5} + \frac{17}{5} J_{0,0,0,2,2/5} \right. \\ & \left. + \frac{1}{4} J_{0,0,0,0,4,2/5} - \frac{6}{5} J_{0,0,1,1,1,2/5} \right) = \frac{77}{10} \sqrt{2\pi} 2^{1/5} \Gamma\left(\frac{16}{5}\right) C_1^2 \\ & = \frac{77}{10} \omega_1. \quad (\text{B31}) \end{aligned}$$

We do not give the expressions for the other few coefficients  $\mathcal{B}_k$  in terms of the basic integrals, since they are too much cumbersome to be written explicitly. Computations of these is straightforward, and yields the results

$$\mathcal{B}_3 = \frac{1}{8} \sqrt{2\pi}, \quad (\text{B32})$$

$$\mathcal{B}_5 = \frac{1806}{250} \omega_0, \quad (\text{B33})$$

$$\mathcal{B}_6 = \frac{567}{12500} \omega_0, \quad (\text{B34})$$

$$\mathcal{B}_8 = \frac{149054}{13750} \omega_1, \quad (\text{B35})$$

$$\mathcal{B}_9 = \frac{348424}{5500000} \omega_1. \quad (\text{B36})$$

### APPENDIX C: CALCULATIONS OF THE BASIC INTEGRALS $J_{k,l,m,n,p,\alpha}$

In this appendix we give some details for the calculations of the basic integrals  $J_{k,l,m,n,p,\alpha}$ . We need only integrals for  $n=0, 1$ , and  $2$ . Evaluation of the integral for  $n=0$  is straightforward; however, for  $n=1$  and  $2$  it requires some tricks which are described, e.g., in Ref. [23].

For  $n=1$  the basic integrals may be written as

$$J_{k,l,m,1,p,\alpha} = \int d\vec{g} \int d\vec{C} \phi(g) \phi(C) C^k g^l (\vec{C} \cdot \vec{g})^m (\vec{C} \cdot \vec{I}(g)), \quad (\text{C1})$$

with  $\vec{g} \equiv \vec{c}_{12}$ , and with the vectorial integral

$$\vec{I}(g) \equiv \int d\mu \vec{e} |\vec{g} \cdot \vec{e}|^\alpha (\vec{g} \cdot \vec{e})^p, \quad (\text{C2})$$

with the short-hand notation  $d\mu = d\vec{e} \Theta(-\vec{g} \cdot \vec{e}) |\vec{g} \cdot \vec{e}|$ .

Similarly, for  $n=2$  one can write

$$J_{k,l,m,2,p,\alpha} = \int d\vec{g} \int d\vec{C} \phi(g) \phi(C) C^k g^l (\vec{C} \cdot \vec{g})^m \vec{C} \cdot \hat{H}(g) \cdot \vec{C}, \quad (\text{C3})$$

where the dyad  $\hat{H}(g)$  is given by

$$\hat{H}(g) \equiv \int d\mu \vec{e} \circ \vec{e} |\vec{g} \cdot \vec{e}|^\alpha (\vec{g} \cdot \vec{e})^p, \quad (\text{C4})$$

and where  $\circ$  denotes a direct vector product. Due to symmetry one can write  $\vec{I}(g) = \vec{g} G(g)$ , where the function  $G(g)$  may be found from the equation

$$\vec{g} \cdot \vec{I}(g) = g^2 G(g) = \int d\vec{e} \Theta(-\vec{g} \cdot \vec{e}) |\vec{g} \cdot \vec{e}|^{1+\alpha} (\vec{g} \cdot \vec{e})^{p+1}. \quad (\text{C5})$$

The integral on the right-hand side of Eq. (C5) may be evaluated using spherical coordinates:

$$\int d\vec{e} \Theta(-\vec{g} \cdot \vec{e}) |\vec{g} \cdot \vec{e}|^\beta (\vec{g} \cdot \vec{e})^r = \frac{2\pi(-1)^r}{r+\beta+1} g^{r+\beta}. \quad (\text{C6})$$

This yields the function  $G(g)$  and, thus, the vectorial integral

$$\vec{I}(g) = 2\pi(-1)^{p+1} \frac{g^{p+\alpha}}{p+\alpha+3} \vec{g}. \quad (\text{C7})$$

For the dyad  $\hat{H}(g)$  one can also use symmetry arguments to write

$$\hat{H}(g) = A(g) \vec{g} \circ \vec{g} + B(g) g^2 \hat{U}, \quad (\text{C8})$$

where  $\hat{U}$  is a unit dyad (i.e., a diagonal matrix). Multiplying  $\hat{H}$  from both sides by  $\vec{g}$  and then taking a trace, we obtain the set of equations for the functions  $A(g)$  and  $B(g)$ :

$$\begin{aligned} \vec{g} \cdot \hat{H} \cdot \vec{g} &= A g^4 + B g^4 = \int d\vec{e} \Theta(-\vec{g} \cdot \vec{e}) |\vec{g} \cdot \vec{e}|^{1+\alpha} (\vec{g} \cdot \vec{e})^{p+2} \\ &= \frac{2\pi(-1)^p}{p+\alpha+4} g^{p+\alpha+3} \end{aligned} \quad (\text{C9})$$

and

$$\begin{aligned} \text{Tr} \hat{H} &= A g^2 + 3B g^2 = \int d\vec{e} \Theta(-\vec{g} \cdot \vec{e}) |\vec{g} \cdot \vec{e}|^{1+\alpha} (\vec{g} \cdot \vec{e})^p \\ &= \frac{2\pi(-1)^p}{p+\alpha+2} g^{p+\alpha+1}. \end{aligned} \quad (\text{C10})$$

Solving the set of equations (C9) and (C10) for  $A(g)$  and  $B(g)$ , we obtain

$$\hat{H} = \frac{2\pi(-1)^p g^{p+\alpha-1}}{(p+\alpha+4)(p+\alpha+2)} [(p+\alpha+1) \vec{g} \circ \vec{g} + g^2 \hat{U}]. \quad (\text{C11})$$

With Eqs. (C7) and (C11) the basic integrals  $J_{k,l,m,n,p,\alpha}$  for  $n=1$  and  $2$  can be reduced to the integrals

$$\int d\vec{g} \int d\vec{C} \phi(g) \phi(C) C^{k+\nu_1} g^{l+p+\alpha+\nu_2} (\vec{C} \cdot \vec{g})^{m+\nu_3}, \quad (\text{C12})$$

with  $\nu_1=0$ ,  $\nu_2=0$ , and  $\nu_3=1$  to evaluate the integral for  $n=1$ , and with  $\nu_1=0$ ,  $\nu_2=-1$ , and  $\nu_3=2$  and  $\nu_1=2$ ,  $\nu_2=1$ , and  $\nu_3=0$  for  $n=2$ . The computation of these integrals is straightforward, and yield the final results (B19), (B20), and (B21) given above.

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