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Contact of Viscoelastic Spheres

Thomas Schwager and Thorsten Pöschel [a]
Humboldt-Universität zu Berlin, Institut für Physik,
Invalidenstraße 110, D-10115 Berlin, Germany

In a recent paper an implicit equation for contacting viscoelastic spheres was derived [1]. Integrating this equation it can be shown that the coefficient of normal restitution ϵ depends on the impact velocity g as $1-\epsilon \sim g^{\frac{1}{5}}$

The behavior of granular gases has been of large scientific interest in recent time. Goldhirsch and Zanetti [2] and McNamara and Young [3] have shown that a homogeneous granular gas is unstable. After some time one observes dense regions (clusters) and voids. To evaluate the loss of mechanical energy due to collisions one introduces the coefficient of (normal) restitution

$$g' = -\epsilon g \tag{1}$$

describing the loss of relative normal velocity g' of a pair of colliding particles after the collision with respect to the impact velocity g

In the investigations [2,3] the approximation of constant coefficient of restitution was assumed. Solving viscoelastic equations for spheres currently it was shown that the coefficient of normal restitution ϵ is not a constant but a function of the impact velocity $\epsilon\,(g)$ itself[1,4]. For the "compression" $\xi = R_1 + R_2 - |\vec{r}_1 - \vec{r}_2|$ of particles with radii R_1 and R_2 at positions $\vec{r}_1$ and $\vec{r}_2$ one finds

$$\ddot{\xi} + \rho \left(\xi^{3/2} + \frac{3}{2}\, A \,\sqrt{\xi}\, \dot{\xi} \right) = 0 \tag{2}$$

$$\rho = \frac{2\; Y \sqrt{R_{\text{eff}}}}{3\; m_{\text{eff}}\,(1-\nu^2)} \tag{3}$$

$$m_{\text{eff}} = \frac{m_1 m_2}{m_1 + m_2} \qquad R_{\text{eff}} = \frac{R_1 R_2}{R_1 + R_2}$$

Y is the Young modulus and m_{eff} and R_{eff} are the effective radius and mass of the grains, respectively. A is a material constant depending on the Young modulus, the

[a] schwager@summa.physik.hu-berlin.de, thorsten@itp02.physik.hu-berlin.de

viscous constants and the Poisson ratio of the material. The initial conditions for solving (2) are $\xi(0) = 0$ and $\dot{\xi}(0) = g$. The coefficient of restitution ϵ of at time $t = 0$ colliding spherical grains can be found from this equation relating the relative normal velocities $g = \dot{\xi}(0)$ at time of impact and at time t_c, when the particles separate after the collision, i.e. t_c is the collision time:

$$\epsilon = -\dot{\xi}\left(t_c\right)/\dot{\xi}\left(0\right). \tag{4}$$

The (numerical) integration of equation (4) yields the coefficient of restitution as a function of the impact velocity (see fig. 1 in ref. [1]) which is in good agreement with experimental data[5]. Constant coefficient of restitution, however, does *not* agree with experimental experience [6]. Other theoretical work on this topic can be found e.g. in [7,8].

The duration of collision t_c^0 for the undamped problem ($A = 0$) is given by [9]

$$t_c^0 = \frac{\Theta_c^0}{\rho^{\frac{2}{5}} g^{\frac{1}{5}}} \;, \tag{5}$$

with Θ_c^0 being a constant. Substituting $\xi = \rho^{-2}x(\Theta)$ we get for the rescaled velocity $v = \rho^2 g$, and with (5) one finds $t = \Theta\, v^{-\frac{1}{5}}$. We rewrite (2) using rescaled compression, relative velocity and time x, v and Θ and the abbreviation $\alpha = \frac{3}{2}A$

$$\ddot{x} + \alpha v^{-\frac{1}{5}} \dot{x}\sqrt{x} + v^{-\frac{2}{5}} x^{\frac{3}{2}} = 0 \tag{6}$$

with $\dot{x} = \frac{d}{d\Theta}x$. The initial conditions read now

$$x(0) = 0 \tag{7}$$

$$\frac{dx}{dt}(0) = v^{\frac{1}{5}} \frac{dx}{d\Theta}(0) = v, \quad \text{hence} \quad \dot{x}(0) = v^{\frac{4}{5}}. \tag{8}$$

Eq. (6) will be solved now by series expansion. All derivatives of third order and higher of x diverge at $\Theta = 0$, hence one cannot expand x in powers of Θ. Since ξ can be written in the form $\xi(t) = g\,t\,y(t)$, the Ansatz

$$x(\Theta) = v^{\frac{4}{5}}\Theta\ (1 + \eta(\Theta)) \quad \text{with} \quad \eta(0) = 0 \tag{9}$$

seems to be a reasonable assumption. We get an equation for η :

$$\Theta\ddot{\eta} + 2\dot{\eta} + \alpha v^{\frac{1}{5}} \Theta^{\frac{3}{2}} \dot{\eta}\sqrt{1+\eta} + \left(\alpha v^{\frac{1}{5}}\sqrt{\Theta} + \Theta^{\frac{3}{2}}\right)(1+\eta)^{\frac{3}{2}} = 0\ . \tag{10}$$

In (10) occur terms $\Theta^{0.5}$ and $\Theta^{1.5}$, therefore we expand η in powers of $\sqrt{\Theta}$

$$\eta = \sum_{k=0}^{\infty} a_k\ \Theta^{\frac{k}{2}} \tag{11}$$

The first coefficients $a_0 = 0$ and $a_1 = 0$ vanish because of the initial conditions. With Taylor expansion of $\sqrt{1+\eta}$ and $(1+\eta)^{\frac{3}{2}}$ for small η we arrive at

$$\eta = -\frac{4}{15}\alpha v^{\frac{1}{5}}\Theta^{\frac{3}{2}} - \frac{4}{35}\Theta^{\frac{5}{2}} + \frac{3}{70}\alpha v^{\frac{1}{5}}\Theta^4 + \frac{1}{15}\alpha^2 v^{\frac{2}{5}}\Theta^3 \ldots \tag{12}$$

and therefore

$$x = v^{\frac{4}{5}}\ \Theta - \frac{4}{15}\alpha v\ \Theta^{\frac{5}{2}} - \frac{4}{35}v^{\frac{4}{5}}\Theta^{\frac{7}{2}} + \frac{1}{15}\alpha^2 v^{\frac{6}{5}}\Theta^4$$
$$+\frac{3}{70}\alpha v\Theta^5 - \frac{38}{2475}\alpha^3 v^{\frac{7}{5}}\Theta^{\frac{11}{2}} + \frac{1}{175}v^{\frac{4}{5}}\Theta^6 + \dots \tag{13}$$

Rearranging the full series (13) one finds (s. fig. 1)

$$x = v^{\frac{4}{5}}x_0(\Theta) + \alpha v x_1(\Theta) + \alpha^2 v^{\frac{6}{5}}x_2(\Theta)\dots \tag{14}$$

$v^{\frac{4}{5}}x_0$ is the solution of the undamped (elastic) collision. We will need $x\left(\frac{1}{2}\Theta_c^0\right)$ where Θ_c^0 is the duration of the undamped collision.

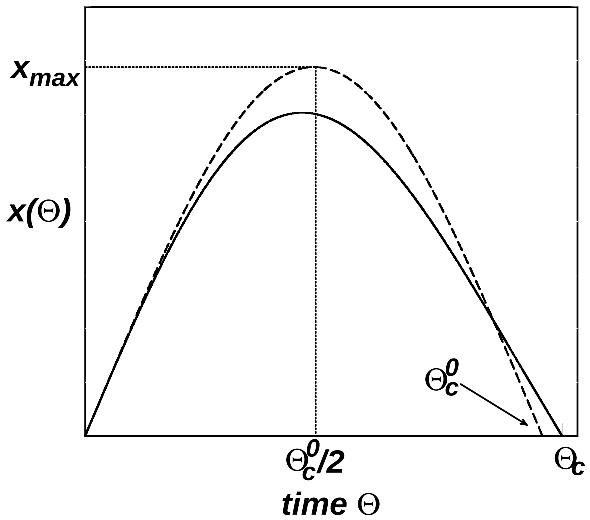

FIG. 1.: The dynamics of the collision. The dashed line shows the (strictly symmetric) solution of the undamped collision. For the case of the damped motion (full line) the maximum penetration depth is achieved earlier whereas the duration of the collision is longer ($\Theta_c > \Theta_c^0$). The figure gives a nice impress why we employ the *inverse collision* instead of direct calculation: one cannot expand x beyond Θ_c^0. In the point Θ_c^0 the derivatives of the curve for elastic motion diverge.

Later we will need the solution x^{inv} of the *inverse problem*, i.e. of a collision with impact velocity v' and final velocity v. Hence the inverse collision is not a damped motion, but an accelerated one, where α has to be replaced by $-\alpha$. Substituting $v \to v'$ and $\alpha \to -\alpha$ we find

$$x^{inv}(\Theta') = (v')^{\frac{4}{5}}x_0(\Theta') - \alpha v' x_1(\Theta') + \alpha^2 (v')^{\frac{6}{5}}x_2(\Theta')\dots \tag{15}$$

Now we determine the collision time Θ_c and the final velocity $\frac{d}{dt}x\left(\Theta_c\right)$. (A more direct method to calculate Θ_c is to determine the solution of $x(\Theta) = 0$ using Taylor expansion of x in the region close to Θ_c^0. It can be seen easily that this method fails since, all derivatives of $\frac{\partial^n x}{\partial \Theta^n}$ with $n \geq 3$ diverge for $\Theta = \Theta_c^0$. Therefore Θ_c has to be calculated indirectly as explained in the text (s. fig. 1).)

The problem will be subdivided into two parts (s. fig. 2): a) the motion of the particles x from $\Theta = 0$ to time Θ_m when x approaches its maximum and where $\dot{x}$ changes its sign, and b) from Θ_m to Θ_c. In case of undamped motion where $\alpha = 0$ we have $\Theta_m = \Theta_c^0/2$. In part b) we do not consider the collision itself but the inverse problem in the interval $(\Theta = 0,\ \Theta_m')$, with Θ_m' being the time where x^{inv} approaches its maximum. The continuity of both parts means

$$x\left(\Theta_m\right) = x^{inv}\left(\Theta_m'\right) \ . \tag{16}$$

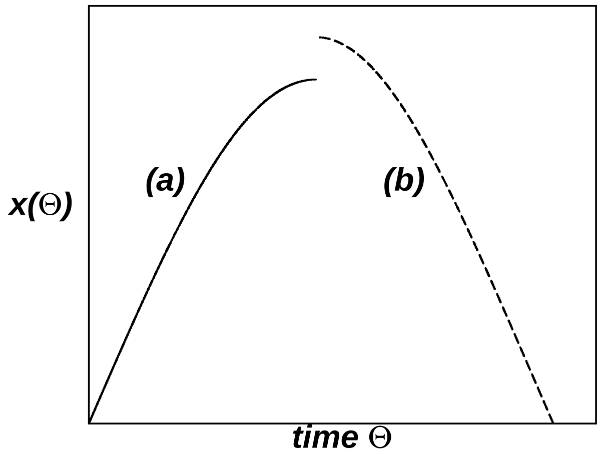

FIG. 2.: Sketch of the calculation. The first part (a) $\Theta \in (0, \Theta_m)$ is calculated directly, for the other part (b) we define the *inverse collision* where the particles start with velocity v' and velocity approaches zero at $\Theta = \Theta_m$. Both curves have to fit together smoothly.

For finite damping $\alpha \neq 0$ we write $\Theta_m = \Theta_c^0/2 + \delta$ and $\Theta_m' = \left(\Theta_c^0\right)'/2 + \delta'$ and remind that $\Theta_c^0 = \left(\Theta_c^0\right)'$. To get an expression for δ we expand

$$\dot{x}\left(\frac{\Theta_c^0}{2} + \delta\right) = 0 = \dot{x}\left(\frac{\Theta_c^0}{2}\right) + \delta\ddot{x}\left(\frac{\Theta_c^0}{2}\right) + \frac{\delta^2}{2}\frac{d^3}{d\Theta^3}x\left(\frac{\Theta_c^0}{2}\right) + \dots \tag{17}$$

$$= v^{\frac{4}{5}}\left(\dot{x}_0\left(\frac{\Theta_c^0}{2}\right) + \delta\ddot{x}_0\left(\frac{\Theta_c^0}{2}\right) + \frac{\delta^2}{2}\frac{d^3}{d\Theta^3}x_0\left(\frac{\Theta_c^0}{2}\right) + \dots\right) \tag{18}$$

$$+v\alpha\left(\dot{x}_1\left(\frac{\Theta_c^0}{2}\right) + \delta\ddot{x}_1\left(\frac{\Theta_c^0}{2}\right) + \frac{\delta^2}{2}\frac{d^3}{d\Theta^3}x_1\left(\frac{\Theta_c^0}{2}\right) + \dots\right) \tag{19}$$

and using $\dot{x}_0\left(\Theta_c^0/2\right) = 0$ ($v^{\frac{4}{5}}\ x_0$ is the solution of the undamped problem)

$$\delta = -\alpha v^{\frac{1}{5}}\frac{\dot{x}_1\left(\Theta_c^0/2\right)}{\ddot{x}_0\left(\Theta_c^0/2\right)} + \mathcal{O}\left(\alpha^2\right) \ . \tag{20}$$

The expression (20) has to be inserted into the Taylor expansion of $x\left(\Theta_c^0/2 + \delta\right)$:

$$
\begin{aligned}
x\left(\Theta_c^0/2+\delta\right) &= v^{\frac{4}{5}}\left(x_0\left(\frac{\Theta_c^0}{2}\right)+\delta\dot{x}_0\left(\frac{\Theta_c^0}{2}\right)+\frac{\delta^2}{2}\ddot{x}_0\left(\frac{\Theta_c^0}{2}\right)+\dots\right)\\
&\quad+\alpha v\left(x_1\left(\frac{\Theta_c^0}{2}\right)+\delta\dot{x}_1\left(\frac{\Theta_c^0}{2}\right)+\frac{\delta^2}{2}\ddot{x}_1\left(\frac{\Theta_c^0}{2}\right)+\dots\right) \qquad (21)\\
&= v^{\frac{4}{5}}x_0\left(\frac{\Theta_c^0}{2}\right)+\alpha v x_1\left(\frac{\Theta_c^0}{2}\right)-\frac{\alpha^2 v^{\frac{6}{5}}}{2}\frac{\dot{x}_1^2\left(\Theta_c^0/2\right)}{\ddot{x}_0\left(\Theta_c^0/2\right)}\\
&\quad+\alpha^2 v^{\frac{6}{5}}x_2\left(\frac{\Theta_c^0}{2}\right)+\mathcal{O}\left(\alpha^3\right)\ . \qquad (22)
\end{aligned}
$$

Hence

$$
\begin{aligned}
x\left(\Theta_m\right) &= v^{\frac{4}{5}}x_0\left(\Theta_c^0/2\right)+\alpha v x_1\left(\Theta_c^0/2\right)+\\
&\quad \alpha^2 v^{\frac{6}{5}}\left(x_2\left(\frac{\Theta_c^0}{2}\right)-\frac{1}{2}\frac{\dot{x}_1^2\left(\Theta_c^0/2\right)}{\ddot{x}_0\left(\Theta_c^0/2\right)}\right)+\dots \qquad (23)
\end{aligned}
$$

Replacing again $v\to v'$ and $\alpha\to-\alpha$ yields

$$
\begin{aligned}
\delta' &= \alpha(v')^{\frac{1}{5}}\frac{\dot{x}_1\left(\Theta_c^0/2\right)}{\ddot{x}_0\left(\Theta_c^0/2\right)}+\mathcal{O}\left(\alpha^2\right) \qquad (24)\\
x^{inv}\left(\Theta_m'\right) &= (v')^{\frac{4}{5}}x_0\left(\Theta_c^0/2\right)-\alpha v' x_1\left(\Theta_c^0/2\right)+\\
&\quad \alpha^2 (v')^{\frac{6}{5}}\left(x_2\left(\frac{\Theta_c^0}{2}\right)-\frac{1}{2}\frac{\dot{x}_1^2\left(\Theta_c^0/2\right)}{\ddot{x}_0\left(\Theta_c^0/2\right)}\right)+\dots \qquad (25)
\end{aligned}
$$

As explained above both solutions (23) and (25) have to be equal. With

$$
\beta = x_2\left(\frac{\Theta_c^0}{2}\right)-\frac{1}{2}\frac{\dot{x}_1^2\left(\Theta_c^0/2\right)}{\ddot{x}_0\left(\Theta_c^0/2\right)} \qquad (26)
$$

we write

$$
\begin{aligned}
v^{\frac{4}{5}}x_0\left(\frac{\Theta_0^c}{2}\right)+\alpha v x_1\left(\frac{\Theta_0^c}{2}\right)+\alpha^2 v^{\frac{6}{5}}\beta &= (v')^{\frac{4}{5}}x_0\left(\frac{\Theta_0^c}{2}\right)-\alpha v' x_1\left(\frac{\Theta_0^c}{2}\right)\\
&\quad+\alpha^2 (v')^{\frac{6}{5}}\beta\ . \qquad (27)
\end{aligned}
$$

We expand v' in α

$$
v' = v+\alpha v_1+\alpha^2 v_2+\dots\ , \qquad (28)
$$

and find

$$
\begin{aligned}
&v^{\frac{4}{5}}x_0\left(\frac{\Theta_0^c}{2}\right)+\alpha v x_1\left(\frac{\Theta_0^c}{2}\right)+\alpha^2 v^{\frac{6}{5}}\beta\\
&= v^{\frac{4}{5}}\left(1+\frac{\delta v}{v}\right)^{\frac{4}{5}}x_0\left(\frac{\Theta_0^c}{2}\right)-\alpha v\left(1+\frac{\delta v}{v}\right)x_1\left(\frac{\Theta_0^c}{2}\right)+\alpha^2 v^{\frac{6}{5}}\left(1+\frac{\delta v}{v}\right)^{\frac{6}{5}}\beta \qquad (29)
\end{aligned}
$$

with $\delta v = \alpha v_1 + \alpha^2 v_2 + \ldots$. Writing $(1+\frac{\delta v}{v})^{\frac{n}{5}}$ in powers of α and comparing coefficients yields finally

$$\begin{aligned} v' &= v\left(1+\frac{5}{2}\alpha v^{\frac{1}{5}}\frac{x_1\left(\frac{\Theta_c^0}{2}\right)}{x_0\left(\frac{\Theta_c^0}{2}\right)}+\frac{15}{4}\alpha^2 v^{\frac{2}{5}}\left(\frac{x_1\left(\frac{\Theta_c^0}{2}\right)}{x_0\left(\frac{\Theta_c^0}{2}\right)}\right)^2+\ldots\right) \\ &= v\left(1-\alpha v^{\frac{1}{5}}C_1+\alpha^2 v^{\frac{2}{5}}C_2+\ldots\right) \end{aligned} \tag{30}$$

and for coefficient of normal restitution one gets

$$\epsilon = \frac{v'}{v} = 1-\alpha v^{\frac{1}{5}}C_1+\alpha^2 v^{\frac{2}{5}}C_2+\ldots \tag{31}$$

$$= 1-\frac{3}{2}C_1 A\rho^{\frac{2}{5}}g^{\frac{1}{5}}+\frac{9}{4}C_2 A^2\rho^{\frac{4}{5}}g^{\frac{2}{5}}+\ldots \tag{32}$$

with g being the impact velocity (fig 3). For the duration of collision we find with (20), (25) and (30)

$$t_c = \Theta_c^0\rho^{-\frac{2}{5}}g^{\,-\frac{1}{5}}\left(1+\frac{1}{10}C_1\alpha\rho^{\frac{2}{5}}g^{\frac{1}{5}}\right)+\mathcal{O}(\alpha^2)\ . \tag{33}$$

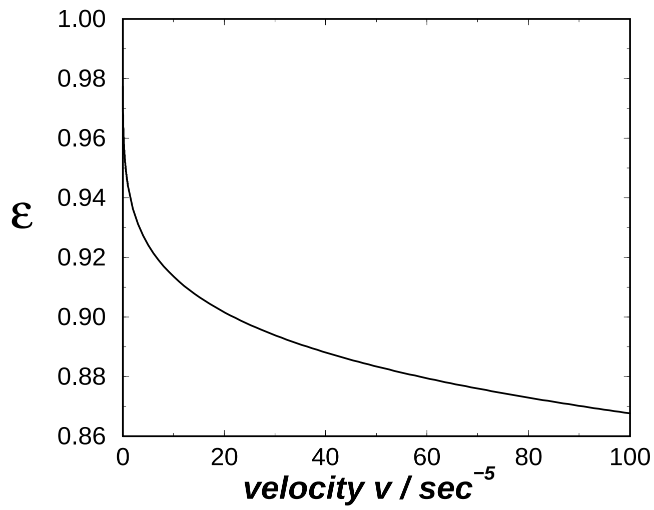

FIG. 3.: The coefficient of restitution over impact velocity due to eq. (32). As expected for small relative velocity the particles collide almost elastically. The result of numerical integration of (4) coincides with the curve. Both curves cannot be distinguished in the plot.

Our final results eq. (32) shows that for viscoelastic colliding smooth bodies the coefficient of normal restitution is a decreasing function with rising impact velocity: $1-\epsilon \sim g^{\frac{1}{5}}$.

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