Can we scale granular systems?

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ABSTRACT: For the experimental investigation of large scale phenomena in the laboratory such as in geophysical or industrial applications one has to scale down all length in the system, e.g. particle size, container size. We show that besides length scaling one as to scale the material properties too to achieve identical behavior of the scaled and the original systems. We provide the scaling laws for a system of viscoelastic spheres.

1 INTRODUCTION

In many cases granular systems cannot be investigated in their original size but they have to be scaled to meet the restrictions of the laboratory size. Scaling all lengths of the system such as particle sizes, container geometry etc. by a constant factor α may turn big boulders to centimeter sized particles and helps to reduce the costs of experimental investigations considerably. Of course, one desires that the effects which occur in the original system, occur equivalently in the scaled system too. We will show that naive scaling will modify the properties of a granular system such that the original system and the scaled system might reveal quite different dynamical properties. To guarantee equivalent dynamical properties of the original and the scaled systems we have to modify the material properties in accordance with the scaling factor and we have to redefine the unit of time. The appropriate scaling laws depend on the particular grain model. Here we investigate viscoelastic interaction between contacting particles.

In a simple approximation, a granular system may be described as an assembly of spheres of radii R_i, $i = 1, \ldots, N$. If two particles i and j at positions $\vec{r}_i$ and $\vec{r}_j$ of radii R_i and R_j touch, i.e., if $\xi_{ij} \equiv R_i + R_j - |\vec{r}_i - \vec{r}_j| > 0$ they feel the force

$$\vec{F}_{ij} = F^n_{ij}\vec{n}_{ij} + F^t_{ij}\vec{t}_{ij}\,, \tag{1}$$

with the unit vector in normal direction $\vec{n} \equiv (\vec{r}_j - \vec{r}_i)/|\vec{r}_j - \vec{r}_i|$ and the respective unit vector in tangential direction $\vec{t}$.

The dynamics of the system can be found by numerically integrating the equations of motion for all particles $i = 1, \ldots, N$ simultaneously with appropriate initial conditions. Eventually, external forces as, e.g. gravity, may also act on the particles. The detailed formulation of the normal and tangential forces F^n and F^t depends on the grain model (e.g. (Schäfer, Dippel, and Wolf 1996)).

Assume we know the dynamics of a certain granular system S. Will the dynamics change if we rescale all sizes by a constant factor α, i.e., $R_i^\prime \equiv \alpha R_i$, but leaving the material properties constant? Here and in the following we mark all variables which describe the scaled system $S^\prime$ with a prime. If the scaling affects the system properties, how do we have to modify the material properties to assure that the systems S and $S^\prime$ behave identically?

2 THE NORMAL FORCE F^n

The normal force F^n can be subdivided into the elastic and the dissipative part, $F^n = F^n_{\text{el}} + F^n_{\text{dis}}$. (For simplicity of notation we omit the indices ij of the variables which refer to pairs of particles.)

The elastic force for colliding spheres is given by Hertz's law (Hertz 1882)

$$F^n_{\text{el}} = \frac{2Y}{3\left(1-\nu^2\right)}\sqrt{R^{\text{eff}}}\xi^{3/2} \equiv \rho\xi^{3/2}\,, \tag{2}$$

with $R^{\text{eff}} = R_iR_j/\left(R_i + R_j\right)$ and Y, ν being the Young modulus and the Poisson ratio. Eq. (2) also defines the prefactor ρ which we will need below.

The formulation of the dissipative part of the force F^n_{dis} depends on the mechanism of damping. The most simple mechanism is viscoelastic damping, which we will focus on. Other mechanisms as plastic deformation or brittle deformation are more complicated since the shape of the particles changes due to collisions.

Therefore, the simultaneous assumption of plastic deformation and spherical shape of the particles is inconsistent (although frequently applied in simulations of granular material).

Viscoelastic interaction implies that the elastic part of the stress tensor is a linear function of the deformation tensor and the dissipative part of the stress tensor is a linear function of the deformation rate tensor. The viscoelastic interaction assumption is valid if the characteristic velocity (the impact rate g) is much smaller than the speed of sound c in the material and the viscous relaxation time $\tau_{\rm vis}$ is much smaller than the duration of the collisions τ_c. The range of our assumption is, hence, limited from both sides: the collisions should not be too fast to assure $g \ll c$, $\tau_{\rm vis} \ll \tau_c$, and not too slow to avoid influences of surface effects as adhesion. For spheres the dissipative force reads (Brilliantov, Spahn, Hertzsch, and Pöschel 1996)

$$F^n_{\rm dis} = A \frac{d\xi}{dt}\frac{d}{d\xi}F^n_{\rm el} = \frac{3}{2}A\rho\sqrt{\xi}\,\frac{d\xi}{dt}\,, \tag{3}$$

$$\text{with} \quad A = \frac{1}{3}\frac{\left(3\eta_2-\eta_1\right)^2}{3\eta_2+2\eta_1}\frac{\left(1-\nu\right)\left(1-2\nu\right)}{Y\nu^2}\,. \tag{4}$$

The dissipative material constant A is a function of the viscous constants $\eta_{1/2}$, the Young modulus Y and the Poisson ratio ν. The functional form of Eq. (3) was guessed (but not strictly derived) before (Kuwabara and Kono 1987) and has been derived independently (Brilliantov, Spahn, Hertzsch, and Pöschel 1996) and (Morgado and Oppenheim 1997) using very different approaches. However, only the strict analysis of the viscoelastic deformation by Brilliantov et al. yields the prefactors A and ρ as functions of the material properties. The knowledge of the functional dependence of these prefactors is crucial for the derivation of the scaling properties.

Combining the forces (2) and (3) yields the equation of motion for interacting viscoelastic spheres

$$\frac{d^2\xi}{dt^2} + \frac{\rho}{m^{\rm eff}}\left(\xi^{3/2} + \frac{3}{2}A\sqrt{\xi}\,\frac{d\xi}{dt}\right) = 0\,, \tag{5}$$

with $m^{\rm eff} = m_i m_j/(m_i+m_j)$ and with the boundary conditions $\xi|_{t=0} = 0$ and $d\xi/dt|_{t=0} = g$.

3 SCALING PROPERTIES

If two particles interacting via the force (2) collide with relative velocity g their maximal compression is

$$\xi_0 \equiv \left(\frac{5}{4}\frac{m^{\rm eff}}{\rho}\right)^{2/5} g^{4/5}\,, \tag{6}$$

which can be derived by equating the kinetic energy of the impact $m^{\rm eff}g^2/2$ with the elastic energy at the instant of maximal compression $2\rho\xi_0^{5/2}/5$, neglecting the influence of dissipation. The length ξ_0 can be considered as a characteristic length of the system. As characteristic time we define the time in which the distance between the particles changes by the characteristic length ξ_0 just before the collision starts:

$$\tau_0 \equiv \xi_0/g\,. \tag{7}$$

Note that up to a numerical prefactor the timescale τ_0 is equal to the duration of the undamped collision (Hertz 1882), which would be an alternative (and equivalent) choice of the timescale. The definitions Eqs. (6,7) allow to write the equation of motion (5) in dimensionless coordinates (Ramírez, Pöschel, Brilliantov, and Schwager 1999)

$$\hat{\xi} \equiv \frac{\xi}{\xi_0}, \qquad \frac{d\hat{\xi}}{d\tau} = \frac{1}{g}\frac{d\xi}{dt}, \qquad \frac{d^2\hat{\xi}}{d\tau^2} = \frac{\xi_0}{g^2}\frac{d^2\xi}{dt^2}$$

$$\frac{d^2\hat{\xi}}{d\tau^2} + \frac{5}{4}\hat{\xi}^{3/2} + \frac{3}{2}\left(\frac{5}{4}\right)^{3/5} A\left(\frac{\rho}{m^{\rm eff}}\right)^{2/5} g^{1/5}\sqrt{\hat{\xi}}\,\frac{d\hat{\xi}}{d\tau} = 0\,, \tag{8}$$

with boundary conditions $\hat{\xi}\Big|_{\tau=0} = 0$ and $d\hat{\xi}\Big/d\tau\Big|_{\tau=0} = 1$.

We see that the only term which depends explicitly on the system size and on material properties is the prefactor in front of the third term of Eq. (8). A scaled system, therefore, can only be equivalent to the unscaled one if this term remains unchanged.

Expanding our abbreviations we obtain

$$A\left(\frac{\rho}{m^{\rm eff}}\right)^{2/5} g^{1/5} = A\left(\frac{2Y\sqrt{R^{\rm eff}}}{3(1-\nu^2)m^{\rm eff}}\right)^{2/5} g^{1/5} \tag{9}$$

$$\sim AY^{2/5}\phi^{-2/5}\left(1-\nu^2\right)^{-2/5} F(R_i,R_j)\,g^{1/5}, \tag{10}$$

with ϕ being the material density. The function $F(R_i,R_j)$ in Eq. (10) collects all terms containing R_i and R_j. Eq. (10) equals Eq. (9) up to the constant $(2\pi)^{-2/5}$ which is not relevant for the scaling.

The function $F(R_i,R_j)$ is directly affected by scaling the radii $R_i^\prime = \alpha R_i$, $R_j^\prime = \alpha R_j$. One can easily check that its scaling is

$$F(R_i^\prime,R_j^\prime) = \alpha^{-1}F(R_i,R_j)\,. \tag{11}$$

4 SCALING LARGE PHENOMENA DOWN TO "LAB-SIZE" EXPERIMENTS

The previous expressions show that scaling all lengths by a factor α affects the prefactor (9) already via the scaling properties of $F\left(R_i,R_j\right)$. To guarantee identical behavior of the original and the scaled systems we have to modify the material properties in a way to assure that the equations of motion of both systems are equivalent, which in turn assures that the prefactors

(9) of the original system and the scaled system are identical. We are going to derive the necessary scaling of the material properties in this section.

One of the few things which cannot be modified in an experiment with reasonable effort is the constant of gravity G. That implies that going from S to S' not only G but all other accelerations must remain unaffected too. Therefore, we require

$$\left(\frac{d^2x}{dt^2}\right)' = \frac{d^2(\alpha x)}{d\,(t')^2} = \frac{d^2x}{dt^2}\,, \quad \text{i.e.,} \quad t' = \sqrt{\alpha}\,t\,. \quad (12)$$

Hence, scaling all lengths $x' = \alpha x$ implies that times scale as $t' = \sqrt{\alpha} t$. Thus, our clock in the laboratory should run by a factor $\sqrt{\alpha}$ faster or slower than the clock in the original system. In other words, if in the original system we observe a phenomenon at time $t = 100$ sec, we will find the same effect in the scaled system at time $t' = \sqrt{\alpha}\,100$ sec. Scaling of time is a direct consequence of the spatial scaling if the constant of gravity has the same value in both systems.

In the scaled system the equation of motion reads

$$\frac{d^2\xi'}{dt'^2} + \frac{\rho'}{(m^{\text{eff}})'}\,(\xi')^{3/2} + A'\,\frac{\rho'}{(m^{\text{eff}})'}\sqrt{\xi'}\,\frac{d\xi'}{dt'} = 0\,. \quad (13)$$

If we apply our scaling relations as introduced above, i.e. $\xi' = \alpha\xi$, $d\xi'/dt' = \sqrt{\alpha}\,(d\xi/dt)$ and $d^2\xi'/dt'^2 = d^2\xi/dt^2$, we obtain

$$\frac{d^2\xi}{dt^2} + \alpha^{3/2}\frac{\rho'}{(m^{\text{eff}})'}\xi^{3/2} + \alpha A'\,\frac{\rho'}{(m^{\text{eff}})'}\sqrt{\xi}\,\frac{d\xi}{dt} = 0\,. \quad (14)$$

This has to be equal to the unscaled equation of motion (5) yielding the conditions to assure the dynamical equivalence of both systems:

$$\frac{\rho'}{(m^{\text{eff}})'} = \alpha^{-3/2}\frac{\rho}{m^{\text{eff}}} \quad \text{and} \quad A' = \sqrt{\alpha}\,A\,, \quad (15)$$

and after inserting the definition of the elastic parameter ρ we find

$$\frac{Y'}{\phi'\left(1-(\nu')^2\right)} = \alpha\frac{Y}{\phi(1-\nu^2)} \quad \text{and} \quad A' = \sqrt{\alpha}\,A\,. \quad (16)$$

5 THE TANGENTIAL FORCE F^t

The tangential force, of course, must scale in the same way as the normal force, or gravity respectively. From Eq. (15) we see that

$$\frac{(F^n)'}{(m^{\text{eff}})'} = \frac{\rho'}{(m^{\text{eff}})'}(\xi')^{3/2} = \alpha^{-3/2}\left(\frac{\rho}{m^{\text{eff}}}\right)\alpha^{3/2}\xi^{3/2} \quad (17)$$

and thus find that the normal force scales as the mass of the particles

$$(F^n)'\Big/\left(m^{\text{eff}}\right)' = F^n\big/m^{\text{eff}}\,. \quad (18)$$

Therefore, we only have to require

$$\left(F^t\right)'\Big/\left(m^{\text{eff}}\right)' = F^t\big/m^{\text{eff}}\,. \quad (19)$$

This condition has to be met by appropriately scaling the material constants, in particular the constant of tangential friction, resulting in an additional scaling equation. Its form depends on the underlying friction model. If we assume the most simple law

$$F^t = \mu F^n\,, \quad (20)$$

with μ being the friction coefficient we find the friction coefficient being invariant with respect to scaling

$$\mu' = \mu\,. \quad (21)$$

Other friction models will lead to different scaling properties. In general, the discussion of the scaling of the tangential force is a more complicated issue which has to be discussed in detail elsewhere.

6 NUMERICAL SIMULATION

In the previous sections we have shown that for a system of viscoelastic spheres naive scaling of lengths changes the dynamics of the system. To demonstrate the derived scaling laws which assure identical behavior of the original and the scaled systems we perform a simulation of 1000 particles of average particle diameter 10 cm on an inclined surface of length 10 m and slope 15 degrees. The scaling factor is chosen to be $\alpha = 1/16$ (I), which leads to a scaled system of length 62.5 cm, and average particle diameter 6.25 mm, where each particle has been individually resized, and $\alpha = 16$ (II) leading to a system of length 160 m and of average particle diameter of 160 cm. Velocities will therefore scale with a factor 1/4 (I) and 4 (II) as well as time scales. The material parameters are summarized in the table below. Particles are subjected to normal viscoelastic forces as well as tangential forces (of the simple type discussed above) which, although not very realistic, are enough for the purpose of demonstrating the scaling properties.

samples	original	scaled (I)	scaled (II)
ϕ	2 g/cm^3	2 g/cm^3	2 g/cm^3
$\dfrac{Y}{\left(1-\nu^2\right)}$	8 GPa	0.5 GPa	128 GPa
A	$2\cdot 10^{-5}$ s	$5\cdot 10^{-6}$ s	$8\cdot 10^{-5}$ s
μ	0.1	0.1	0.1

We have also performed simulations on the systems I and II where only the lengths have been scaled but the material parameters were kept unchanged (wrong scaling). The difference between the correct and wrong scaling can be appreciated in Fig. 1, where we show average velocity and density profiles of stationary flow under periodic boundary conditions.

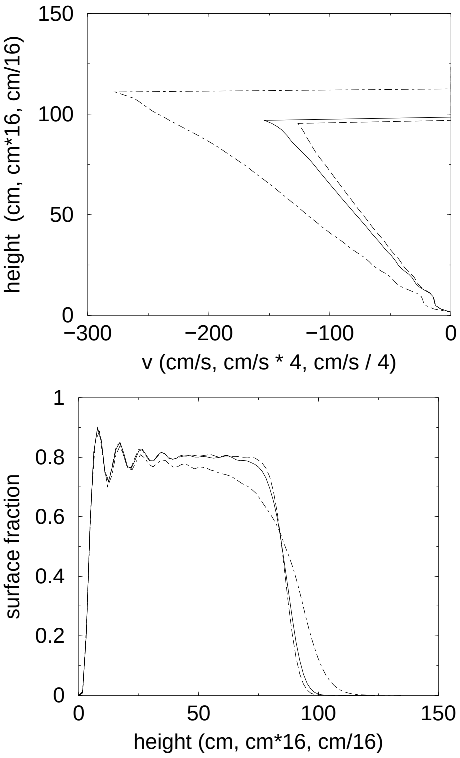

Figure 1: Velocity and density profiles of the flow down an inclined plane, for scaled systems I and II (see text). Solid line: original and scaled systems with scaled material properties as specified in the table (the data collapse to one single line). Dashed line: system I where only lengths have been scaled, but not material parameters (wrong scaling). Dot-dashed line: system II with wrong scaling. Velocity and length axes have been appropriately scaled for each sample.

7 CONCLUSIONS

We have shown that by means of a scaling procedure one can construct for any given system of viscoelastic spheres an equivalent system of scaled size having the same dynamic properties. Along with a change of size of the system and all of its constituents by a factor α, one has to modify the timescale (rate of the internal clock) and the material properties (see table).

	original system	scaled system
all lengths	x	$\alpha\, x$
time	t	$\sqrt{\alpha}\, t$
elastic const.	$\dfrac{Y}{\phi\left(1-\nu^2\right)}$	$\alpha\, \dfrac{Y}{\phi\left(1-\nu^2\right)}$
dissip. const.	A	$\sqrt{\alpha} A$

If one scales an experiment by a factor α, therefore, one has to change the material as well according to the scaling relations given in the table, in order to find the same effects as in the original system. Moreover one has to scale time, i.e. an effect which is observed at time t in the original system will occur at time $\sqrt{\alpha}\, t$ in the rescaled system. We want to give an example: Assume in the original system one deals with steel spheres ($Y = 20.6 \cdot 10^{10}$ Nm^{-2}, $\nu = 0.29$ and $\phi = 7,700$ kg m^{-3}) of average radius $\bar{R} = 10$ cm and system size of $L = 10$ m, hence $Y/(\phi(1-\nu^2)) = 2.92 \cdot 10^7$ m^{2}sec^{-2}. One wishes to measure a certain value at time $t = 100$ sec. In the lab we perform the experiment with an equivalent system of size $L' = 1$ m, i.e. we scale the system by $\alpha = 0.1$, including all radii. From the scaling relations we see that we have to find a material with $Y'/(\phi'(1-\nu'^2)) \approx 0.3 \cdot 10^7$ m^{2}sec^{-2}. From tables (Kuchling 1989) we see that we can use plexiglass ($Y = 0.32 \cdot 10^{10}$ Nm^{-2}, $\nu = 0.35$ and $\phi = 1,200$ kg m^{-3}) in order to obtain this value. Therefore, we have to perform the experiment with plexiglass spheres and have to measure the value of interest at time $t' = 31.6$ sec.

One can imagine that not for all scaling factors α one will find a proper material, however, nowadays it is possible to manufacture materials which can meet demanding requirements, such as high softness along with a custom-designed damping constant.

This scaling scheme has a number of useful applications. The most notable one is the possibility to scale down real world systems, e.g. geophysical or industrial granular systems, to sizes where laboratory experiments can be performed.

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