Rolling as a "continuing collision"

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Abstract. We show that two basic mechanical processes, the collision of particles and rolling motion of a sphere on a plane, are intimately related. According to our recent findings, the restitution coefficient for colliding spherical particles ϵ, which characterizes the energy loss upon collision, is directly related to the rolling friction coefficient μ_{roll} for a viscous sphere on a hard plane. We quantify both coefficients in terms of material constants which allows to determine either of them provided the other is known. This relation between the coefficients may give rise to a novel experimental technique to determine alternatively the coefficient of restitution or the coefficient of rolling friction.

Is the collision of two spheres related to rolling motion of a sphere on a plane? Probably the answer will be "No", if the idealised version of these two basic processes (*i.e.* absolutely elastic collision and ideal rolling without any resistance) is assumed. However, in reality neither ideal collision nor ideal rolling occur since in both processes mechanical energy is lost according to dissipative material deformation. Quantifying the losses a profound similarity of these two processes has been revealed. It results in a relation between quantities describing energy loss due to collision and rolling and provides a novel view on rolling which may be considered as a continuing collision.

It is well-known since Newton's times that two balls colliding with velocities v_1 and v_2 have smaller after-collisional velocities v_1' and v_2' as compared with the perfectly elastic collision. The decrease of the particle velocity or in other words the loss of mechanical energy is described by the coefficient of restitution ϵ. For a pair of identical particles colliding on a line one has:

$$v_1' = v_1 - \frac{1+\epsilon}{2} g,$$
$$v_2' = v_2 + \frac{1+\epsilon}{2} g \qquad (1)$$

where $g = v_1 - v_2$ is the initial relative velocity. The ideal elastic collision is described by $\epsilon = 1$, so that $v_1' = v_2$ and $v_2' = v_1$. The relative velocity $g = v_1 - v_2$ changes at the collision as

$$g' = -\epsilon g. \qquad (2)$$

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We want to remark that for the more general case of a collision equations (1, 2) hold true but the pre- and after-collisional velocities are vectors $\mathbf{v}_{1/2}$, $\mathbf{v}_{1/2}'$ while $\mathbf{g}$ and $\mathbf{g}'$ are the pre- and after-collisional relative normal velocities

$$\mathbf{g} = [(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{e}] \, \mathbf{e} \qquad (3)$$
$$\mathbf{g}' = [(\mathbf{v}_1' - \mathbf{v}_2') \cdot \mathbf{e}] \, \mathbf{e}$$

where $\mathbf{e} = (\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|$ with $\mathbf{r}_{1/2}$ being the particle positions at the instant of collision.

The non-ideal character of the collision originates from the *dissipative* force

$$F_{\mathrm{diss}} = \frac{3}{2} A \, \rho \, \sqrt{\xi} \dot{\xi} \qquad (4)$$

acting on the colliding spheres which has been derived recently [1–3]. Here, ξ is the time-dependent compression of the particles during the collision

$$\xi = 2R - |\mathbf{r}_1 - \mathbf{r}_2| \qquad (5)$$

with R being the radius of the spheres, see Figure 1. The material constant ρ depends on the Young modulus Y and Poisson ratio ν of the particle material as

$$\rho = \frac{2Y}{3(1 - \nu^2)} \sqrt{R/2}. \qquad (6)$$

The constant A is expressed in terms of the elastic and viscous material constants [1]. The dissipative force always acts against the relative velocity $\dot{\xi}$ of the particles, so that elastic energy stored during the collision is not completely reconverted into kinetic energy after the collision. In linear approximation with respect to the dissipative parameter
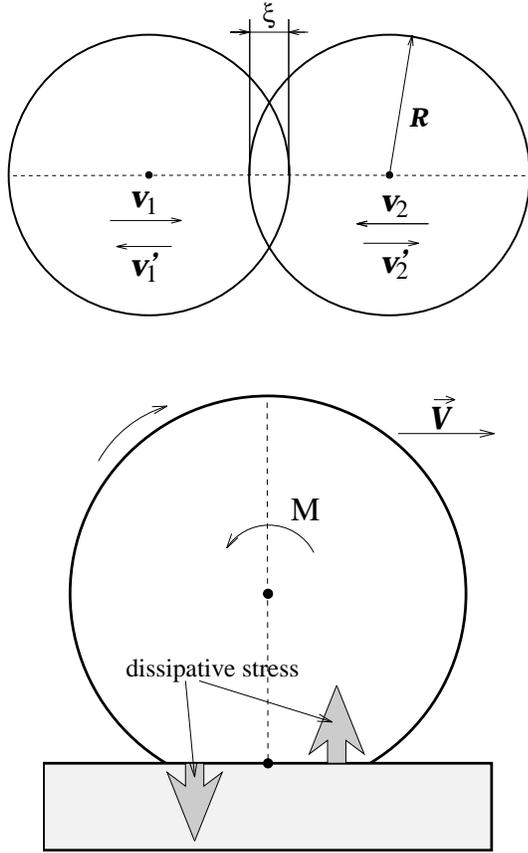
Fig. 1. Sketches of two colliding spheres and of a rolling sphere. Origination of the rolling friction moment due to the dissipative stress is shown.

A the solution of the collision problem which accounts for the dissipative force yields for the restitution coefficient [4]

$$1 - \epsilon = C_1 \left(\frac{3A}{2}\right)\left(\frac{2\rho}{m}\right)^{2/5} g^{1/5} \pm \cdots \qquad (7)$$

with [4,5]

$$C_1 = \frac{\Gamma(3/5)\sqrt{\pi}}{2^{1/5}5^{2/5}\Gamma(21/10)} = 1.15344. \qquad (8)$$

$\Gamma(x)$ is the Gamma-function and m is the mass of the (identical) particles. Hence, the restitution coefficient for viscoelastic collisions depends sensitively on the normal component of the relative velocity g as described by equation (7).

Each collision of particles certainly terminates after some time if attractive forces are excluded, but one can ask, whether some "collision" can proceed permanently.

In spite of being far from apparent one can show that the mechanics of a rolling sphere on a hard plane is intrinsically similar to that of a collision [6] provided the main part of the dissipation originates from viscoelastic bulk deformations of the sphere [7–9]. Indeed, when a soft sphere rolls we notice the same sequence of compression and subsequent decompression as in the collision process.

Thus, in rolling motion, which one can treat as a "continuing" collision, we observe a steady dissipation due to incomplete retransformation of elastic energy during decompression. This results in a rolling friction moment M, which acts against the motion:

$$M = \mu_{\text{roll}} F^N \qquad (9)$$

(see Fig. 1). Here, F^N is the normal force exerted by the plane onto the sphere caused by the sphere's own weight. Calculations performed for a soft sphere rolling on a hard (undeformable) plane reveal that

$$\mu_{\text{roll}} = AV, \qquad (10)$$

where V is the sphere's linear velocity and A is exactly the *same* material constant as in the law of collision [6]. The relation (10) has been obtained for the case when the deformation of the rolling sphere is small compared to its radius, the velocity V is much less than the speed of sound in the material, and when the relaxation time of the rolling process, estimated as the ratio of the sphere deformation and the velocity V is much larger than the dissipative relaxation times of the viscoelastic material [6].
Thus,

$$\frac{1 - \epsilon}{b\,(\rho/m)^{2/5}\,g^{1/5}} = \frac{\mu_{\text{roll}}}{V} \qquad (11)$$

with

$$b = 3C_1/2^{3/5} = 2.28296 \qquad (12)$$

relates the rolling friction coefficient μ_{roll} to the coefficient of normal restitution ϵ for identical particles colliding with relative velocity g.

The linear equation (11) refers to the case of small impact velocities of the colliding spheres. This situation is the most favourable for performing experimental studies. One can, however, apply a more general nonlinear relation which accounts for high-order corrections of the dissipative parameter A. This follows from the high-order expansion for the restitution coefficient [5]:

$$\begin{aligned}
1 - \epsilon = {} & C_1 \left(\frac{3A}{2}\right)\left(\frac{2\rho}{m}\right)^{2/5} g^{1/5} \\
& - C_2 \left(\frac{3A}{2}\right)^2 \left(\frac{2\rho}{m}\right)^{4/5} g^{2/5} \\
& + C_3 \left(\frac{3A}{2}\right)^3 \left(\frac{2\rho}{m}\right)^{6/5} g^{3/5} \\
& - C_4 \left(\frac{3A}{2}\right)^4 \left(\frac{2\rho}{m}\right)^{8/5} g^{4/5} \pm \cdots \qquad (13)
\end{aligned}$$

where [5]

$$\begin{aligned}
C_2 &= 3/5\, C_1^2 \\
C_3 &= 0.315119\, C_1^3 \\
C_4 &= 0.161167\, C_1^4.
\end{aligned}$$

Introducing then

$$Z \equiv b \left(\frac{\rho}{m}\right)^{2/5} g^{1/5} \frac{\mu_{\text{roll}}}{V} \tag{14}$$

one can write the generalization of the linear equation (11) as

$$\frac{1-\epsilon}{Z} = 1 - 0.6\,Z + 0.315119\,Z^2 - 0.161167\,Z^3 \pm \cdots \tag{15}$$

Physically the relations (11, 15) are based on the intrinsic mechanical similarity of the collision and rolling processes. In practice equation (11) (or (15)) may be used to determine either of coefficients ϵ or μ_{roll} provided the other one is known.

The measurement of restitution coefficients for spheres is a complicated experimental problem, in particular if one is interested in collisions at very low impact velocity, *e.g.* [10]. These values are of great importance in several problems, *e.g.* for the description of the kinetics of planetary ring material where the particles typically collide with velocities of the range $\left(10^{-2} \ldots 10^{-3}\right)$ m/s. The derived relation between the coefficient of rolling friction and the coefficient of normal restitution allows to determine the latter value by measuring the resistance of a sphere against rolling on a hard plane which might be experimentally less complicated.

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