Scaling properties of granular materials

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Abstract. Given an assembly of viscoelastic spheres with certain material properties, we raise the question how the macroscopic properties of the assembly will change if all lengths of the system, i.e. radii, container size etc., are scaled by a constant. The result leads to a method to scale down experiments to lab-size.

1 Introduction

It is frequently desired to investigate large scale phenomena in granular systems experimentally, such as geophysical effects or industrial applications. To this end one has to rescale all lengths of the system to a size which is in accordance with the laboratory size, i.e. big boulders in the original system become centimeter sized particles in the experiment. Of course, one wishes that the effects which occur in the large system occur equivalently in the scaled system too. We will show that naive scaling will modify the properties of a granular system such that the original system and the scaled system might reveal quite different dynamic properties. To guarantee equivalent dynamical properties of the original and the scaled systems we have to modify the material properties in accordance with the scaling factor and we have to redefine the unit of time.

In a simple approximation, a granular system may be described as an assembly of spheres of radii $R_i$, $i = 1, \ldots, N$. The interaction between spheres $i$ and $j$ at positions $r_i$ and $r_j$ of radii $R_i$ and $R_j$ is given by a pairwise force law

$$F_{ij} = \begin{cases} F_{ij}^n n_{ij} + F_{ij}^t t_{ij} & \text{if } \xi_{ij} = R_i + R_j - |r_i - r_j| > 0 \\ 0 & \text{else} \end{cases}$$

(1)

with the unit vector in normal direction $n \equiv \frac{r_j - r_i}{|r_j - r_i|}$ and the respective unit vector in tangential direction $t$. 
The dynamics of the system can be found, in principle, by integrating Eq. (1) for all particles \(i = 1, \ldots, N\) simultaneously with appropriate initial conditions. Eventually, external forces as, e.g., gravity, may also act on the particles. In practice the dynamics is found by integrating Eq. (1) numerically. Molecular Dynamics techniques (e.g. [1]) exploit this idea, which has provided much insight into the properties of granular systems over the last decade (see e.g. [2] and many references therein).

The detailed formulation of the normal and tangential forces \(F^n\) and \(F^t\) depends on the grain model; several models have been studied in [3]. It is known that the tangential force \(F^t\) is essential to explain many macroscopic effects, in particular if static properties of the granular material become important [4].

With this formulation we mean that at least for a certain time interval the static hindrance with respect to rotation of contacting spheres is important for the dynamics of the system. Nevertheless there exist phenomena for which the rotational degree of freedom of the particles is less important as, e.g., in the case of highly agitated granular system as granular gases [5]. In other systems neglecting the tangential force \(F^t\) still might be a good approximation. In the present study we first assume that the tangential force \(F^t\) may be neglected to describe the dynamics of the system, i.e., \(F_{ij} = F^n n_{ij}\). The incorporation of tangential forces is discussed briefly in section 6.

Assume we know the dynamics of a certain granular system \(S\). Will the dynamics change if we rescale all sizes by a constant factor \(\alpha\), i.e., \(R_i' = \alpha R_i\), but leaving the material properties constant? Here and in the following we mark all variables which describe the scaled system \(S'\) with a prime. If the scaling affects the system properties, how do we have to modify the material properties to assure that the systems \(S\) and \(S'\) behave identically?

### 2 The normal force \(F^n\)

The normal force \(F^n\) can be subdivided into two parts, the elastic part \(F^n_{el}\) and the dissipative part \(F^n_{dis}\):

\[
F^n = F^n_{el} + F^n_{dis}. \tag{2}
\]

(For simplicity of notation we drop the indices \(ij\) of the variables which refer to pairs of particles.)

The elastic force for colliding spheres is given by Hertz’s law [6]

\[
F^n_{el} = \frac{2Y}{3(1-\nu^2)} \sqrt{R^\text{eff}} \xi^{3/2} = \rho \xi^{3/2}, \tag{3}
\]

with \(R^\text{eff} = R_i R_j / (R_i + R_j)\) and \(Y, \nu\) being the Young modulus and the Poisson ratio. Equation (3) also defines the prefactor \(\rho\) which we will need below.

The formulation of the dissipative part of the force \(F^n_{dis}\) depends on the mechanism of damping. The most simple mechanism is elastic interaction (\(F^n_{dis} = 0\)). The second simplest type is viscoelastic damping, which we will focus on. More complicated mechanisms are plastic deformation or brittle deformation. The last
two damping types are more complicated since the shape of the particles changes due to collisions. Therefore, the simultaneous assumption of plastic deformation and spherical shape of the particles is inconsistent (although frequently applied in simulations of granular material). It is not even clear that the spherical shape of particles which undergo plastic deformation is conserved on average [7].

We assume the most simple nontrivial interaction between colliding grains, which is viscoelastic interaction. It implies that the elastic part of the stress tensor is a linear function of the deformation tensor, whereas the dissipative part of the stress tensor is a linear function of the deformation rate tensor. It is valid if the characteristic velocity (the impact rate \( g \)) is much smaller than the speed of sound \( c \) in the material and the viscous relaxation time \( \tau_{vis} \) is much smaller than the duration of the collisions \( \tau_c \) (quasistatic motion).

The range of our assumption is, hence, limited from both sides: the collisions should not be to fast to assure \( g \ll c, \tau_{vis} \ll \tau_c \), and not too slow to avoid influences of surface effects as adhesion. For spheres the dissipative part of the normal force reads [8]

\[
F^n_{\text{dis}} = A \frac{d}{dt} \frac{d}{dt} F^n_{\text{el}} \\
= \frac{3}{2} A \rho \sqrt{\xi} \frac{d\xi}{dt}
\]  

(4)

(5)

with

\[
A = \frac{1}{3} \left( 3\eta_2 - \eta_1 \right)^2 \frac{(1 - \nu)(1 - 2\nu)}{3\eta_2 + 2\eta_1 Y \nu^2}.
\]

(6)

The dissipative material constant \( A \) is a function of the viscous constants \( \eta_{1/2} \), the Young modulus \( Y \) and the Poisson ratio \( \nu \). The functional form of Eq. (5) was guessed (but not derived) by Kuwabara and Kono before [9] and has been derived independently by Brilliantov et al. [8] and Morgado and Oppenheim [10] using very different approaches. However, only the strict analysis of the viscoelastic deformation in Ref. [8] yields the prefactors \( A \) and \( \rho \) as functions of the material properties. The knowledge about these prefactors is crucial for the derivation of the scaling properties. We want to remark that the general equation (4) is not limited to spheres but holds true for any smooth (in mathematical sense) interaction of viscoelastic bodies [11].

Combining the forces (3) and (5) one obtains the equation of motion for colliding viscoelastic spheres

\[
\frac{d^2 \xi}{dt^2} + \frac{\rho}{m_{\text{eff}}} \left( \xi^{3/2} + \frac{3}{2} A \sqrt{\xi} \frac{d\xi}{dt} \right) = 0
\]

\[
\left. \xi \right|_{t=0} = 0
\]

\[
\left. \frac{d\xi}{dt} \right|_{t=0} = g,
\]

with \( m_{\text{eff}} = m_i m_j / (m_i + m_j) \).
3 Scaling properties

First we want to write down the equation of motion (7) in a dimensionless form \[12\]. To this end we need a characteristic length and a characteristic time of the system. A reasonable inherent length of the problem of colliding viscoelastic spheres is the maximal compression $\xi_0$ for the equivalent undamped (elastic) problem. It can be found by equating the kinetic energy of the impact $m_{\text{eff}} g^2 / 2$ with the elastic energy at the instant of maximal compression:

$$m_{\text{eff}} g^2 / 2 = m_{\text{eff}} \frac{\rho}{\text{m}^{\text{eff}}} \frac{2}{5} \xi_0^{5/2},$$

yielding

$$\xi_0 \equiv \left( \frac{5 m_{\text{eff}}}{4 \rho} \right)^{2/5} g^{4/5}. \quad (8)$$

As characteristic time we define the time in which the distance between the particles changes by the characteristic length just before the collision starts:

$$\tau_0 \equiv \xi_0 / g. \quad (9)$$

Note that up to a numerical prefactor the timescale $\tau_0$ is equal to the duration of the undamped collision \[6\], which would be an alternative (and equivalent) choice of the timescale. Using the definitions (9) and (10) we find the rescaled length, velocity, and acceleration

\[
\begin{align*}
\dot{\xi} &= \xi / \xi_0, \\
\frac{d\dot{\xi}}{d\tau} &= \frac{1}{g} \frac{d\xi}{dt}, \\
\frac{d^2\dot{\xi}}{d\tau^2} &= \frac{\xi_0}{g^2} \frac{d^2\xi}{dt^2}
\end{align*}
\]

and rewrite the equation of motion (7) in dimensionless form

\[
\begin{align*}
\frac{d^2\dot{\xi}}{d\tau^2} + \frac{5}{4} \dot{\xi}^{3/2} + \frac{3}{2} \left( \frac{5}{4} \right)^{3/5} A \left( \frac{\rho}{m_{\text{eff}}} \right)^{2/5} g^{1/5} \sqrt{\xi} \frac{d\dot{\xi}}{d\tau} &= 0 \quad (14) \\
\dot{\xi} \bigg|_{\tau=0} &= 0 \\
\frac{d\dot{\xi}}{d\tau} \bigg|_{\tau=0} &= 1.
\end{align*}
\]

We see that the only term which depends explicitly on the system size and on material properties is the prefactor in front of the third term of Eq. (14). Scaling the system, therefore, can only affect the dynamics of a granular system if it affects the value of this term. In other words, two systems will behave identically (in the scaled variables) if its value is conserved.
Expanding our abbreviations we obtain

\[ A \left( \frac{\rho}{m^{\text{eff}}} \right)^{2/5} g^{1/5} = A \left( \frac{2Y \sqrt{R_0}}{3(1 - \nu^2)m^{\text{eff}}} \right)^{2/5} g^{1/5} \]

\[ = A \left( \frac{Y \sqrt{R_i R_j}}{2\pi (1 - \nu^2) \phi R_i^3 + R_j^3} \right)^{2/5} g^{1/5} \]

\[ \sim AY^{2/5} \phi^{-2/5} (1 - \nu^2)^{-2/5} F(R_i, R_j) g^{1/5} , \]

with \( \phi \) being the material density. The function \( F(R_i, R_j) \) in Eq. (17) collects all terms containing \( R_i \) and \( R_j \). The third line Eq. (17) equals the second line, Eq. (16), up to the constant \( (2\pi)^{-2/5} \) which is not relevant for the scaling properties.

The function \( F(R_i, R_j) \) is directly affected by scaling the radii \( R_i' = \alpha R_i, R_j' = \alpha R_j \). Let us see how this function scales:

\[ F(R_i', R_j') = F(\alpha R_i, \alpha R_j) = \left( \frac{\alpha R_i \alpha R_j}{\alpha^2 R_i^3 + \alpha R_j^3} \right)^{1/5} = \alpha^{-1} \left( \frac{R_i R_j}{R_i^3 + R_j^3} \right)^{1/5} \]

\[ = \alpha^{-1} F(R_i, R_j) . \]

4 Scaling large phenomena down to “lab-size” experiments

We have already seen that simple scaling of the system in general affects the prefactor (15) already via the scaling properties of \( F(R_i, R_j) \) (see Eq. (18)), i.e., in general the original system and the scaled system might reveal quite different dynamic properties. More explicitly, one can show that naively scaling the system by a factor \( \alpha < 1 \) will lead to a comparatively more damped dynamics.

Therefore, to guarantee equivalent dynamical properties of the systems we have to modify the material properties in a way to assure that the prefactors (15) of the original system and the scaled system are identical.

One of the few things which cannot be modified in the experiment with reasonable effort is the constant of gravity \( G \). That implies that going from \( S \) to \( S' \) not only \( G \) but all other accelerations must remain unaffected too:

\[ \left( \frac{d^2 x}{dt^2} \right)' = \frac{d^2 x}{dt^2} \]

\[ \frac{d^2(\alpha x)}{d(t')^2} = \frac{d^2 x}{dt^2} \]

\[ t' = \sqrt{\alpha} t . \]
Hence, scaling all lengths $x' = \alpha x$ implies that times scale as $t' = \sqrt{\alpha} t$ if we request that the gravity constant stays unaffected. Thus, our clock in the laboratory should run by a factor $\sqrt{\alpha}$ faster or slower than the clock in the original system. In other words, if in the original system we observe a phenomenon at time $t = 100 \text{ sec}$, we will find the same effect in the scaled system at time $t' = \sqrt{\alpha} 100 \text{ sec}$. Scaling of time is a direct consequence of the spatial scaling if the constant of gravity has the same value in both systems.

In order to obtain the necessary conditions for the material properties of the scaled system, we require the scaled equation of motion to be exactly equivalent to its counterpart in the unscaled system. This ensures that the trajectories in the scaled system are exactly the same (after reversing the scaling procedure) as in the unscaled system. In the scaled system the equation of motion during a binary collision or a permanent contact reads

$$\frac{d^2 \xi'}{dt'^2} + \frac{\rho'}{(m_{\text{eff}})} \left( \xi' \right)^{3/2} + A' \frac{\rho'}{(m_{\text{eff}})} \sqrt{\xi'} \frac{d \xi'}{dt'} = 0. \quad (20)$$

If we apply our scaling relations which were introduced above, i.e. $\xi' = \alpha \xi$, $\frac{d \xi'}{dt'} = \sqrt{\alpha} \frac{d \xi}{dt}$, $\frac{d^2 \xi'}{dt'^2} = \frac{d^2 \xi}{dt^2}$, we obtain

$$\frac{d^2 \xi}{dt^2} + \alpha^{3/2} \frac{\rho'}{(m_{\text{eff}})} \xi^{3/2} + \alpha A' \frac{\rho'}{(m_{\text{eff}})} \sqrt{\xi} \frac{d \xi}{dt} = 0. \quad (24)$$

Since the scaling does not affect the physical meaning of a given equation of motion, all systems whose equation of motion can be transformed into each other by simply scaling all lengths and the time can be considered to be equivalent. Therefore, our scaled system is equivalent to the unscaled system if and only if

$$\alpha^{3/2} \frac{\rho'}{(m_{\text{eff}})} = \frac{\rho}{m_{\text{eff}}} \quad (25)$$

$$\alpha A' \frac{\rho'}{(m_{\text{eff}})} = A \frac{\rho}{m_{\text{eff}}}. \quad (26)$$

If we choose material constants which obey Eqs. (25) and (26) we will obtain the original equation of motion after scaling the system back to original size, i.e. the two systems are equivalent.

Equations (25) and (26) can be further simplified yielding

$$\frac{\rho'}{(m_{\text{eff}})} = \alpha^{-3/2} \frac{\rho}{m_{\text{eff}}} \quad (27)$$

$$A' = \sqrt{\alpha} A. \quad (28)$$
The last equation shows that $A$, which is essentially the viscous relaxation time of the spheres involved in the contact, has to behave exactly as any other time.

To check the validity of our considerations we will now study two fundamental characteristics of a binary collision whose scaling behaviour is known. These are the coefficient of restitution, which describes the ratio of the normal relative velocity of the two particles after and before the collision

$$\epsilon = \frac{g_{\text{after}}}{g_{\text{before}}},$$

and the duration of the collision. Both values depend on the impact velocity – for experimental evidence see e.g. [13–17]. For viscoelastic spheres the restitution coefficient has been derived rigorously from Eq. (5) [12,18]:

$$\epsilon = 1 - C_1 \frac{3}{2} A \left( \frac{\rho}{m_{\text{eff}}} \right)^{2/5} g^{1/5} + C_2 \left( \frac{3}{2} A \right)^2 \left( \frac{\rho}{m_{\text{eff}}} \right)^{4/5} g^{2/5} + \ldots$$

For the duration of a collision we find [18]

$$T_c = \left( \frac{\rho}{m_{\text{eff}}} \right)^{-2/5} g^{-1/5} \left( D_0 + D_1 \frac{3}{2} A \left( \frac{\rho}{m_{\text{eff}}} \right)^{2/5} g^{1/5} 
+ D_2 \left( \frac{3}{2} A \right)^2 \left( \frac{\rho}{m_{\text{eff}}} \right)^{4/5} g^{2/5} + \ldots \right).$$

To be physically consistent we have to require that the restitution coefficient is invariant and that the duration of the collision scales as any other time, i.e.

$$\epsilon'(g') = \epsilon(g),$$

$$T'_c(g') = \sqrt{\alpha} T_c(g).$$

The symbols $\epsilon'$ and $T'_c$ denote the coefficient of restitution and collision time for spheres in the scaled system, i.e., where the relevant material properties are $\rho', \left( m_{\text{eff}}' \right)$, and $A'$. Eq. (32) is easily verified by noting that

$$A' \left( \frac{\rho'}{m_{\text{eff}}'} \right)^{2/5} (g')^{1/5} = \sqrt{\alpha} \alpha^{-3/5} \alpha^{1/10} A \left( \frac{\rho}{m_{\text{eff}}} \right)^{2/5} g^{1/5}$$

$$ = A \left( \frac{\rho}{m_{\text{eff}}} \right)^{2/5} g^{1/5}. \quad (34)$$

To check Eq. (33) due to property (35) we only have to verify

$$\left( \frac{\rho'}{m_{\text{eff}}'} \right)^{-2/5} (g')^{-1/5} = \sqrt{\alpha} \left( \frac{\rho}{m_{\text{eff}}} \right)^{-2/5} g^{-1/5},$$

which is done by inserting Eqs. (27) and (28).
The discussion of the coefficient of restitution and the duration of collision shows that our scaling procedure is physically consistent. The replacement of our abbreviations with material parameters yields

$$\frac{\rho}{m_{\text{eff}}} = \frac{2Y\sqrt{R_{\text{eff}}}}{3(1-\nu^2)m_{\text{eff}}} = \frac{Y}{2\pi(1-\nu^2)} \left[ F(R_1, R_2) \right]^{5/2}$$  \hspace{1cm} (37)

$$\frac{\rho'}{(m_{\text{eff}})'} = \frac{Y'}{2\pi \left(1-(\nu')^2\right) \phi'} \left[ F(R'_1, R'_2) \right]^{5/2}.$$  \hspace{1cm} (38)

With Eqs. (18), (37) and (38) we obtain

$$\frac{Y'}{(\phi' (1- (\nu')^2))} = \frac{\alpha Y}{\phi (1- \nu^2)}.$$  \hspace{1cm} (39)

Note that in a simple approximation one can identify $\sqrt{Y/\phi}$ with the speed of sound in the material which is strictly valid only for gases. If one neglects the scaling of the Poisson ratio, one discovers that the speed of sound scales as any other velocity, namely with the factor $\sqrt{\alpha}$ (see Eq. (22)).

Hence, simple scaling of lengths by a constant factor $\alpha$ and the request that the gravity constant is conserved leads to a scaling of the elastic and dissipative material properties and of the time if we wish that original and rescaled systems behave identically.

<table>
<thead>
<tr>
<th>all lengths</th>
<th>original system</th>
<th>scaled system</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>$t$</td>
<td>$\sqrt{\alpha} t$</td>
</tr>
<tr>
<td>elastic constant</td>
<td>$Y/\phi (1- \nu^2)$</td>
<td>$\alpha Y/\phi (1- \nu')$</td>
</tr>
<tr>
<td>dissipative constant</td>
<td>$A$</td>
<td>$\sqrt{\alpha} A$</td>
</tr>
</tbody>
</table>

If one scales down an experiment by a factor $\alpha$, therefore, one has to change the material as well according to the scaling relations given in the table, in order to find the same effects as in the original system. Moreover one has to scale time, i.e. an effect which is observed at time $t$ in the original system will occur at time $\sqrt{\alpha} t$ in the rescaled system.

We want to give an example: Assume in the original system one deals with steel spheres ($Y = 20.6 \cdot 10^{10}$ Nm$^{-2}$, $\nu = 0.29$ and $\phi = 7,700$ kg m$^{-3}$) of average radius $R = 10$ cm and system size of $L = 10$ m. The property whose scaling behaviour is known is $Y/(\phi(1-\nu^2)) = 2.92 \cdot 10^{7}$ m$^2$sec$^{-2}$. One wishes to know (to measure) a certain value at time $t = 100$ sec. In the lab we perform the experiment with an equivalent system of size $L' = 1$ m, i.e. we scale the system by the factor $\alpha = 0.1$, including all radii. From the scaling relations we see that we have to find a material whose scaled property is $Y'/(\phi'(1-\nu'^2)) \approx 0.3 \cdot 10^{7}$ m$^2$sec$^{-2}$. From tables [19] we see that we can use plexiglass ($Y = 0.32 \cdot 10^{10}$ Nm$^{-2}$).
Nm$^{-2}$, $\nu = 0.35$ and $\phi = 1,200$ kg m$^{-3}$) in order to obtain this value. Therefore, we have to perform the experiment with plexiglass spheres and have to measure the value of interest at time $t' = 31.6$ sec.

One can imagine that not for all scaling factors $\alpha$ one will find a proper material, however, nowadays it is possible to manufacture materials which can meet demanding requirements, such as high softness along with a custom-designed damping constant.

5 Bouncing ball

One of the most simple experiments one can think of is a ball which falls from height $h$ due to gravity and collides with another ball of the same material which is attached at the ground. For a highly sophisticated experiment on this system see [20]. Assume that in our original system which is to be simulated a steel ball of radius $R = 10$ cm falls from height $h = 20$ cm. The material parameters are $Y = 20.6 \cdot 10^{10}$ Pa, $\phi = 7700$ kg/m$^3$, $\nu = 0.29$, $A = 10^{-4}$ sec, and gravity is $G = 9.81$ m/sec$^2$. Of course, the value of $A$ is fictitious since there are no tabulated values available.

In Fig. 1 we have drawn the distance between the spheres over time for the original system (left) and a system which is scaled down by a factor $\alpha = 0.25$ (right), i.e., the sphere has a radius of $R' = 2.5$ cm and is dropped from $h' = 5$ cm. As can be seen in Fig. 1 both trajectories are virtually indistinguishable if one takes into account that the distance between the balls is scaled by 0.25 and the time by 0.5 in accordance with our scaling scheme.

In Fig. 2 we show the distance between the spheres during the first collision in the unscaled and scaled systems. Negative values mean that the particles are
compressed, i.e. the distance between their centres is smaller than the sum of their radii. Again it can be seen that both systems behave identically.

Fig. 2. The trajectory of the same system as in Fig. 1 during a collision shown with higher resolution in space and time. Again the unscaled system is shown on the left, the scaled one on the right.

6 Consideration of the tangential force

In order to incorporate the tangential force into the analysis we have to require that it scales exactly as the normal force, or gravity respectively. From Eq. (25) we see that

$$\frac{(F_n')}{(m_{\text{eff}}')^\gamma} = \frac{\rho'}{(m_{\text{eff}}')^\gamma} (\xi')^{3/2} = \alpha^{-3/2} \left( \frac{\rho}{m_{\text{eff}}} \right)^{3/2} \xi^{3/2}$$

(40)

and thus find

$$\frac{(F_n')}{(m_{\text{eff}}')^\gamma} = \frac{F_n}{m_{\text{eff}}} \cdot$$

(41)

We see that the normal force scales as the mass of the particles. The same is valid for gravity, which can be seen from the fact that the gravity constant $G$ itself is, by definition, invariant with respect to scaling. To ensure that the scaled system behaves exactly as the unscaled one (taking into account the changed timescale, i.e. the changed rate of the internal clock) we only have to require that

$$\frac{(F_t')}{(m_{\text{eff}}')^\gamma} = \frac{F_t}{m_{\text{eff}}} \cdot$$

(42)

This requirement has to be met by appropriately scaling the material constants, particularly the friction constant. This will give us an additional scaling equation. Its form depends on the underlying friction model, i.e. on the functional dependence of the tangential force on the geometry and the material properties
as well as on the compression and the relative velocity. For instance if we assume the most simple tangential force law

$$F^t = \mu F^n,$$

(43)

with $\mu$ being the friction coefficient we can conclude that the friction coefficient has to be invariant with respect to scaling

$$\mu \equiv \text{const}.$$  

(44)

Other friction models will lead to different scaling properties. However, a thorough discussion of the possible friction models is beyond the scope of this study.

7 Conclusion

In the present article we have shown that by means of a straightforward scaling procedure one can construct for any given system of viscoelastic spheres an equivalent system of scaled size. Along with a change of size of the system and all of its constituents, one has to modify the timescale (rate of the internal clock) and the material properties in a predefined way (see Eqs. (28) and (39)).

This scaling scheme has a number of useful applications. The most notable one is the possibility to scale down real world systems, e.g. geophysical or industrial granular systems, to sizes where laboratory experiments can be performed. If one scales down such a granular system one has to replace the original material by a material which meets the scaling requirements discussed in the text.

One can further apply the scaling scheme in Molecular Dynamics simulations. Here it can be desirable to change the values of the material parameters in order to achieve more accurate results at a given value of the integration time step. A study of the impact of the scaling on the accuracy of numerical simulations is subject of further studies [21].

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