# Nonequilibrium Phase Transition to Anomalous Diffusion and Transport in a Basic Model of Nonlinear Brownian Motion

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(Received 31 March 2021; revised 21 June 2021; accepted 9 August 2021; published 10 September 2021)

We investigate a basic model of nonlinear Brownian motion in a thermal environment, where nonlinear friction interpolates between viscous Stokes and dry Coulomb friction. We show that superdiffusion and supertransport emerge as a nonequilibrium critical phenomenon when such a Brownian motion is driven out of thermal equilibrium by a constant force. Precisely at the edge of a phase transition, velocity fluctuations diverge asymptotically and diffusion becomes superballistic. The autocorrelation function of velocity fluctuations in this nonergodic regime exhibits a striking aging behavior.

DOI: 10.1103/PhysRevLett.127.110601

Nonequilibrium phase transitions are crucial in physics [1], and it was suggested that anomalous diffusion [2] can emerge from such a transition [3]. Nonlinear Brownian motion (NLBM) with friction being a nonlinear function of the particles' velocities [4,5] provides one of the least studied, however, significant physical routes to anomalous diffusion [6–9]. NLBM models emerge, e.g., within out-of-equilibrium physics of cold atomic gases in optical laser-created potentials [6–9], and for active Brownian motion [10–12], where friction and noise are not related by the thermal fluctuation-dissipation theorem (FDT) [13,14]. Below, we investigate whether anomalous diffusion can emerge from a phase transition within another archetype NLBM model obeying FDT [4,5,10,11]:

$$\begin{aligned} \dot{x} &= v \\ m\dot{v} + \eta(v)v &= f(x) + \sqrt{2k_BT\eta(v)}\xi(t). \end{aligned} \tag{1}$$

This model serves as a generalization of the conventional Brownian transport models [4] with linear friction, $\eta(v) = \text{const}$, for the media which respond to perturbations strongly nonlinearly. For example, many nonlinear visco-elastic media [15,16] are characterized by a velocity-dependent viscosity, which leads to nonlinear friction acting on Brownian particles. In Eq. (1), $m$ is the particle mass, $\eta(-v) = \eta(v)$ is a velocity-dependent friction coefficient, $\xi(t)$ is unbiased Gaussian white noise, $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$, $T$ is the temperature of the environment, and $k_B$ is the Boltzmann constant. For the consistency with thermal equilibrium, at $f(x) = 0$, or, in any trapping potential, the interpretation of the multiplicative noise term must be kinetic [5,17,18], also called the Hänggi-Klimontovich interpretation [10]. Then, velocities are Maxwell-Gauss distributed, $p_{st}(v) = \exp[-v^2/(2v_T^2)]/\sqrt{2\pi v_T^2}$ with thermal velocity $v_T = \sqrt{k_BT/m}$, and the energy dissipated by friction is compensated on average by the energy supply from thermal noise such that the averaged kinetic energy of particles is $m\langle v^2\rangle/2 = k_BT/2$, satisfying the classical equipartition theorem. However, beyond equilibrium, when the particles move with some mean velocity $\langle v(t)\rangle$, one can define their kinetic temperature $T_k(t)$ by $m\langle \delta v^2(t)\rangle/2 = k_BT_k(t)/2$, where $\delta v(t) = v(t) - \langle v(t)\rangle$. It is a common notion in classical kinetic theory [19,20] and has the same kinetic meaning as equilibrium temperature. Beyond equilibrium, the kinetic temperature of Brownian particles can largely exceed $T$ [21,22]. Then, one can speak about kinetic heating [23]. Arguably, NLBM is a very general physical model, which, unlike the linear friction model [4], was not studied in sufficient detail.

Differently from *unbiased* NLBM at thermal equilibrium in Refs. [4,5,10,11], we consider below the case of a constant force driving, $f(x) = f = \text{const}$. The corresponding Fokker-Planck equation for the velocity distribution $p(v,t)$ [4,5] can be expressed in a kinetic form

$$\frac{\partial p(v,t)}{\partial t} = \frac{\partial}{\partial v}\left[K(v)e^{-\beta U(v,f)}\frac{\partial}{\partial v}\left(e^{\beta U(v,f)}p(v,t)\right)\right], \tag{2}$$

where $K(v) = v_T^2\eta(v)/m$, by introducing a velocity pseudopotential $U(v,f) = mv^2/2 - mf\int^v dv'/\eta(v')$ and inverse temperature $\beta = 1/(k_BT)$ [25]. The steady-state solution of Eq. (2) reads, $p_{\rm st}(v) \propto \exp[-\beta U(v,f)]$, if it exists being normalizable, which is not necessarily the case for $f \neq 0$ beyond thermal equilibrium and arbitrary $\eta(v)$. However, at $f = 0$, $U(v,0) = mv^2/2$ always and steady-state distribution is thermal for any $\eta(v)$. Next, from $U'_v(v_m,f) = 0$, it follows that the extremal values of velocity $v_m$ in $U(v,f)$ always coincide with the stationary values found from the noiseless dynamics ($T = 0$) in Eq. (1) self-consistently: $v_m = f/\eta(v_m)$. Depending on

 

$\eta(v)$ and the value of $f > 0$, there can be, however, more or less (the minimum stemming from one at $v = 0$, $f = 0$, disappears) than one minimum and also maximum(a) can emerge. It makes the force-and-noise driven NLBM dynamics highly nontrivial and gives, eventually, rise to phase transitions in it.

If $p_{\mathrm{st}}(v)$ exists, any moment of $v$ can be found as

$$\langle v^n(f,T)\rangle_{\mathrm{st}} = \int_{-\infty}^{\infty} v^n e^{-\beta U(v,f)} dv / Z(f,T), \tag{3}$$

where $Z(f,T) = \int_{-\infty}^{\infty} e^{-\beta U(v,f)} dv$ is statistical integral. For $n = 1$, we obtain the mean velocity $\langle v(f,T)\rangle_{\mathrm{st}}$, and the velocity variance can be obtained from (3) as $\langle \delta v^2(f,T)\rangle_{\mathrm{st}} = \langle v^2(f,T)\rangle_{\mathrm{st}} - \langle v(f,T)\rangle_{\mathrm{st}}^2$. It defines the stationary or maximal kinetic temperature $T_{\max}(f) \coloneqq m\langle \delta v^2(f,T)\rangle_{\mathrm{st}}/k_B$ [19,21], which the kinetic temperature $T_k(t)$ can arrive evolving in time. At $f = 0$, $T_{\max}(0) = T$ for any model of nonlinear friction considered. However, for $f > 0$, beyond thermal equilibrium, $T_{\max}$ can be very different from $T$. It can become even infinite within the model considered below.

Next, a general expression for the diffusion coefficient can be found in Ref. [4] and used earlier in Refs. [10–12,25]. For the readers' convenience we provide a concise derivation of this result in Supplemental Material [25,26]. It can be written as

$$D(f,T) = \int_{-\infty}^{\infty} \frac{\psi^2(v) e^{\beta U(v,f)}}{K(v) Z(f,T)} dv, \tag{4}$$

where

$$\psi(v) = -\int_{-\infty}^{v} [v' - \langle v(f,T)\rangle_{\mathrm{st}}] e^{-\beta U(v',f)} dv'. \tag{5}$$

At $f = 0$, Eq. (4) drastically simplifies to

$$D(0,T) = k_B T \left\langle \frac{1}{\eta(v)} \right\rangle_{\mathrm{st}}, \tag{6}$$

where $\langle \eta^{-1}(v)\rangle_{\mathrm{st}} \coloneqq 1/\eta_{\mathrm{eff}}(T)$ is the inverse friction coefficient averaged over equilibrium velocity fluctuations. It can be used to define a temperature-dependent effective friction coefficient $\eta_{\mathrm{eff}}(T)$ in the linear response (LR) regime. This is also a very general result, independently of the model of nonlinear friction $\eta(v)$. $D(f,T)$ is, however, not always finite at $f > 0$. It depends on the model of $\eta(v)$. From Eq. (3) it is easy to show that the linear mobility [4] $\mu(0,T) = \partial\langle v(f,T)\rangle/\partial f|_{f=0} = \eta_{\mathrm{eff}}^{-1}(T) = D(0,T)/(k_B T)$, whenever LR exists.

*Basic model of friction.*—Until this point, the modeling framework and the results are very general. Next, we fix the friction model

$$\eta(v) = \frac{\eta_a}{(1 + v^2/v_c^2)^a}. \tag{7}$$

Here, $\eta_a$ is viscous Stokes friction in the limit $v \ll v_c$, where $v_c$ is a critical velocity. For $0 < a \leq 1$, it corresponds to a popular model of shear-thinning viscosity in nonlinear rheology, the Carreau model [15,27]. For $a = -1$, this is a model considered in Refs. [10,11,28]. For $a = 1$, Eq. (7) corresponds to friction in Refs. [7–9]. We will focus on its special case $a = 1/2$, which interpolates between the viscous friction force $\eta_a v$ for $v \ll v_c$ and the Coulomb-like dry friction $\eta_a v_c v/|v| = f_c v/|v|$ with $f_c = \eta_a v_c$ at $v \gg v_c$. This important model corresponds to a plateau in viscoelastic stress often observed in viscoelastic materials [15,27,29–32] and makes a theoretical model of general interest [33].

In this model,

$$U(v,f) = \frac{m}{2}\Big[v^2 - f v\sqrt{1 + v^2/v_c^2}/\eta_a - f v_c \ln\Big(v/v_c + \sqrt{1 + v^2/v_c^2}\Big)/\eta_a\Big]. \tag{8}$$

It has only one minimum at

$$v_m(f) = \frac{f}{\eta_a\sqrt{1 - (f/f_c)^2}} \tag{9}$$

for a subcritical tilt $f < f_c$ defining the regime of normal transport and diffusion with a steady-state drift $v_m$ at $T = 0$ [34]. For $f \geq f_c$, no $p_{\mathrm{st}}(v)$ exists, both diffusion and transport become anomalously fast. Nonequilibrium phase transition occurs precisely at $f = f_c$, when $Z(f,T)$ diverges. The result comes as a surprise. However, it seems to be a generic feature occurring whenever statistical integral $Z(f,T)$ becomes singular [35]. Also for any $a > 1/2$, no minimum exists for $f$ exceeding some critical value, and, moreover, $Z(f,T)$ is singular for $f > 0$ [40].

Furthermore, zero-force diffusion coefficient (6) can be found in an analytical form using MAPLE: $D(0,T) = D_0 R(T/T_c)$, where $D_0 = k_B T/\eta_a$ is the linear diffusion coefficient, $R(z) = -\sqrt{2/z}\,G_{1,2}^{2,1}(1/(2z)|{}_{0,-1}^{1/2})/(4\pi)$ is expressed in terms of the Meijer $G$ function [41], and $T_c = m v_c^2/k_B$ is a characteristic temperature. For $z \ll 1$, $R(z) \approx 1 + z/2$, $R(1) \approx 1.354530806$, and for $z \gg 1$, $R(z) \approx \sqrt{2z/\pi}$. Hence, at $T \gg T_c$, $D(0,T) \propto T^{3/2}$. The physical range of $f$, where LR is valid can be rather small, when, e.g., $v_c \ll v_T$. Then, it is easy to drive the transport beyond the LR regime.

Below, we study the case $v_T = v_c$ and $T = T_c$, where the LR regime covers a substantial range of forcing, see in Fig. 1, and $v_c$ is used as scaling velocity in nondimensional results. The time is scaled in $\tau_0 = m/\eta_0$, where $\eta_0$ is arbitrary and $\eta_a$ is fixed to $\eta_a = 250$ in these units, with

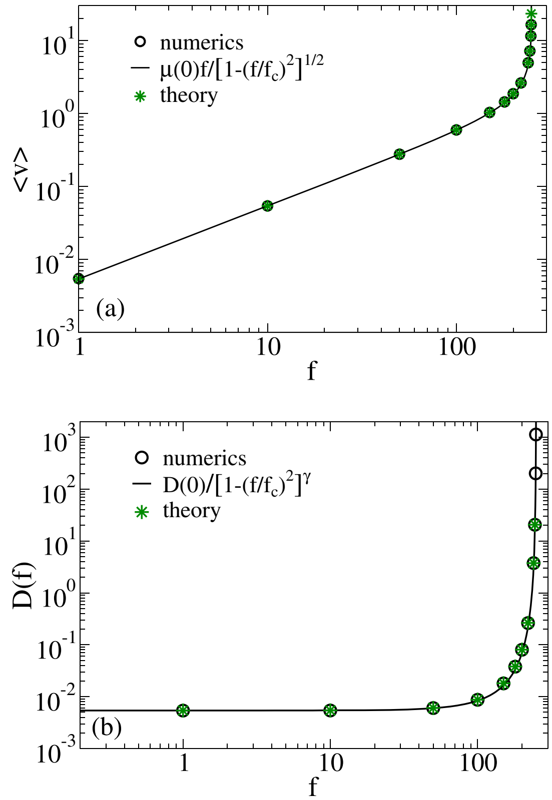


FIG. 1. Numerical results for mean velocity (a) and diffusion coefficient (b) obtained from Langevin simulations (open circles) and analytical theory expressions (green stars), Eqs. (3)–(6). The lines are the fits by simple analytical expressions shown in the plots. At the critical point $f \to f_c$, mean velocity diverges, $\langle v\rangle \propto (f_c - f)^{-1/2}$, with critical exponent $-1/2$, and diffusion coefficient diverges, $D \propto (f_c - f)^{-\gamma}$, with critical exponent $\gamma = 2.537 \pm 0.002$.

$f_c = 250$. The distance is scaled in $x_0 = mv_c/\eta_0$ and energy in $mv_c^2/2$, $T_c = 1$.

*Subcritical normal diffusion and transport regime*.—We first consider the case $f < f_c$. The asymptotical numerical results for mean velocity and diffusion coefficient are shown in Figs. 1(a) and 1(b), correspondingly, where we compare the results obtained from stochastic Langevin simulations (empty circles), see below, and the results obtained from the numerical evaluation of the integrals in Eqs. (3) and (4), and (5) (green stars). They remarkably agree. The mean velocity is very well approximated by Eq. (9), where $1/\eta_a$ is replaced by $\mu(0) = \langle 1/\eta(v)\rangle_{\rm st}$, which is an important result. Furthermore, the numerical results on diffusion are nicely fitted by $D(f) = D(0)/[1 - (f/f_c)^2]^\gamma$, with $D(0)$ from Eq. (6), and fitting exponent $\gamma = 2.537 \pm 0.002$.

*Stochastic simulations*.—In stochastic simulations, we use the stochastic Heun method [26,42]. It corresponds to the Stratonovich intepretation of Eq. (1) [26]. Hence, to have the results consistent with the kinetic interpretation of Eq. (1) and Eq. (2) one must add a spurious drift term $f_{\rm ghost}(v) = (m/2)K'(v)$ [26] to $f(x)$ [26,43]. We did this in simulations, done with the integration time step $\delta t = 2 \times 10^{-5}$ and $N = 10^5$ particles in the ensemble averaging. Velocities are initially thermally distributed and all particles are located at the origin $x = 0$.

*Anomalous diffusion and transport near and at critical tilt*.—It is also important to mention that a long *transient* regime of anomalous diffusion and transport develops when $f$ approaches the edge of phase transition. This regime is characterized a transient kinetic heating [23], $T_k(t) \propto t^{\alpha/2}$, with fitting exponent $\alpha$ approaching $\alpha = 1.336 \pm 0.003$, see in Fig. 2(a), when $f \to f_c^{-0}$. The smaller $f_c - f$, the larger is $T_{\max}(f)$, which is nicely fitted by $T_{\max}(f) = T/(1 - f/f_c)$, see in Fig. 2(c), showing a divergent behavior at the edge of the phase transition to superdiffusion. For ever smaller $f_c - f \ll f_c$, an ever longer intermediate regime of anomalous transport, $\langle \delta x(t)\rangle \propto t^\alpha$, and superballistic diffusion, $\langle \delta x^2(t)\rangle \propto t^{2\alpha}$, emerges [45]. Exactly at $f = f_c$, this transient regime becomes permanent, which is a big surprise. Then, not only $T_{\max}$ diverges but also the autocorrelation time $\tau_{\rm cor} = \int_0^\infty \mathcal{K}_v(\tau, t)d\tau$ of the normalized velocity autocorrelation function $\mathcal{K}_v(\tau, t) = \langle \delta v(t+\tau)\delta v(t)\rangle/\langle \delta v^2(t)\rangle$ considered in the stationary limit $t \to \infty$ diverges. Indeed, in the stationary regime, which still exists at $f < f_c$, $\mathcal{K}_v(\tau, t)$ ceases to depend on the "age" $t$ in the limit $t \to \infty$. Then, the Kubo-Green relation [14] yields $D = \int_0^t \langle \delta v(t+\tau)\delta v(t)\rangle d\tau = \langle \delta v^2\rangle_{\rm st}\tau_{\rm cor} = k_B T_{\max}\tau_{\rm cor}/m$. Next, from the results in Fig. 2(c) we conclude, $\tau_{\rm cor} \propto 1/(1 - f/f_c)^\delta$, with $\delta = \gamma - 1 = 1.537 \pm 0.002$ when $f \to f_c$, i.e., $\tau_{\rm cor}$ diverges. Hence, we are dealing with a genuine nonequilibrium phase transition [1]. Precisely at the point $f = f_c$, $\mathcal{K}_v(\tau, t)$ displays aging behavior [26,48]

$$\mathcal{K}_v(\tau, t) \propto (t/\tau)^{\alpha/2} \tag{10}$$

for $\tau \gg t$ with the same $\alpha$ related to anomalous transport exponent, see Fig. 1(a) in the Supplemental Material [26], where all the curves $\mathcal{K}_v(\tau, t)$ for different $t$ correspond to one and the same master curve upon the rescaling $\tau/t$. Such a behavior is also typical for other complex dynamics exhibiting aging phenomenon [49,50], including anomalous diffusion in nonlinear Brownian dynamics [51,52] and beyond [2,53,54], which are nonergodic [49,51,55]. This explains the origin of a very puzzling regime of nonergodic superballistic diffusion. Indeed, from the model-free [56]

$$\begin{aligned}\langle \delta x^2(t)\rangle &= 2\int_0^t dt' \int_0^{t'} dt'' \langle \delta v(t')\delta v(t'')\rangle \\ &= \frac{2k_B}{m}\int_0^t dt' \int_0^{t'} dt'' T_k(t'')\mathcal{K}_v(t'-t'', t''),\end{aligned} \tag{11}$$

with $T_k(t) \propto t^{\alpha/2}$, cf. Fig. 1(a), and Eq. (10) we immediately obtain $\langle \delta x^2(t)\rangle \propto t^{2\alpha}$, in complete agreement with numerics in Fig. 3.

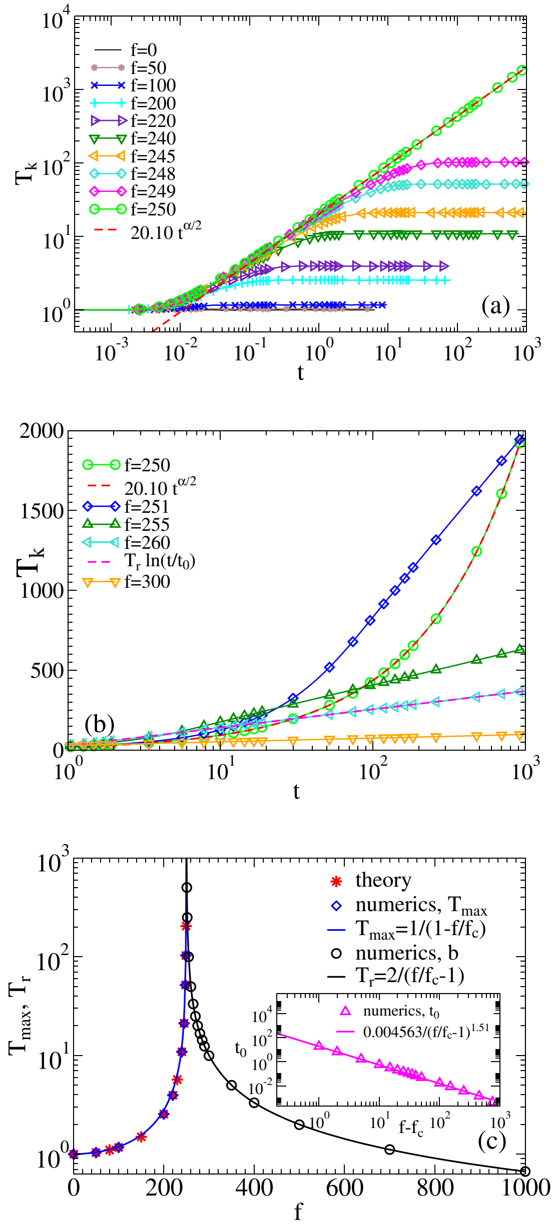


FIG. 2. (a) and (b) Time evolution of kinetic temperature $T_k(t)$ for several values of subcritical (a) and supercritical (b) force $f$, as well as at the critical $f_c$, where $T_k(t) \propto t^{\alpha/2}$ asymptotically with $\alpha = 1.336 \pm 0.003$. For $f < f_c$, $T_k$ eventually saturates arriving at $T_{\max}$, which as a function of $f$ in (c) diverges at the critical point with critical exponent $-1$. For $f > f_c$, $T_k(t) \sim T_r \ln(t/t_0)$ asymptotically diverges in time with $T_r$ and $t_0$ (see inset) in (c) as functions of $f$ also diverging at the critical point. The numerical data are shown by full lines and symbols in (a),(b) and with symbols only in (c). In (c), we depict both Langevin numerics, and for $f < f_c$ also the theoretical result from Eq. (3). Broken lines in (a),(b) and full lines in (c) present various fits shown in the plots.

*Supercritical anomalous diffusion and transport.*— Beyond the point of phase transition, $f > f_c$ fluctuations gradually diminishes with growing $f$. Hence, diffusion becomes slower. However, transport becomes purely ballistic

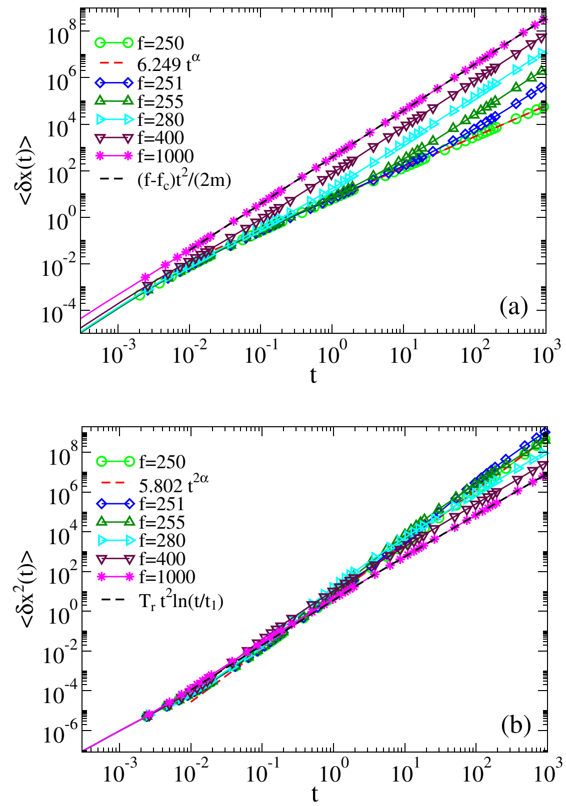


FIG. 3. (a) Mean distance and (b) variance of the particle positions as functions of time for critical and supercritical transport and diffusion, respectively, at several values of force $f$ shown in the plots (full lines with symbols). Dashed lines present various fits discussed in the text.

and nicely described by $\langle \delta x(t) \rangle = (f - f_c)t^2/(2m)$. This is just an acceleration under the force $f - f_c$, as expected from the phase transition to the dry friction at $f = f_c$. Diffusion remains, however, a much more complex, noise-dominated phenomenon. Indeed, the kinetic temperature also grows asymptotically in this case, though logarithmically slow, $T_k(t) \sim T_r \ln(t/t_0)$, with $T_r$ and $t_0$ depending on $f$, see in Fig. 2(c). In this case, asymptotically $\mathcal{K}_v(\tau, t) \approx \mathrm{const} \approx 1$ [48]. It does not decay at all, see Fig. 1(b) in Ref. [26], which is also a clear proof of broken ergodicity by the Slutsky theorem [55]. With these features in Eq. (11) we immediately obtain $\langle \delta x^2(t) \rangle \sim k_B T_r t^2 \ln(t/t_1)/m$ asymptotically with $t_1 = t_0 \exp(3/2)$. The numerics in Fig. 3(b) fully confirm this remarkable relation between $\langle \delta x^2(t) \rangle$ and $T_k(t)$ yielding a weakly superballistic diffusion. We emphasize that also $T_r \approx 2T/(f/f_c - 1)$ and $t_0 \propto 1/(f/f_c - 1)^{\kappa}$ with $\kappa = 1.510 \pm 0.004$ diverge at $f \to f_c^{+0}$, see in Fig. 2(c) confirming the genuine character of nonequilibrium phase transition. Fluctuations $\langle \delta v^2(t) \rangle$ and $\langle \delta x^2(t) \rangle$ are maximized at $f = f_c$ asymptotically. Anomalous diffusion can emerge as a critical phenomenon indeed [3] due to a singularity of statistical integral $Z(f, T)$ in our model at $f \geq f_c$.

In summary, in this Letter we introduced a basic model of nonequilibrium phase transitions in a nonlinear Brownian motion, which is compatible with thermal equilibrium in the absence of external driving and is driven out of equilibrium by a constant force. Within this model class, we discovered and investigated the transition from normal to superdiffusion and supertransport for the friction model interpolating between viscous and dry friction limits. We expect that our results will significant impact not only the general theory of anomalous diffusion and transport but also investigation of anomalous diffusion and transport of various nano- and microparticles in nonlinear viscoelastic media.

Funding of this research by the Deutsche Forschungsgemeinschaft (German Research Foundation), Grant No. GO 2052/3-2 is gratefully acknowledged. We acknowledge support by the Regional Computer Centre Erlangen and Leibniz Supercomputing Centre of the Bavarian Academy of Sciences and Humanities, which kindly provided GPU high-performance computational facilities for doing this work. The work was also supported by the Interdisciplinary Center for Nanostructured Films (IZNF), the Central Institute for Scientific Computing (ZISC), and the Interdisciplinary Center for Functional Particle Systems (FPS) at Friedrich-Alexander University Erlangen-Nürnberg.

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[34] Near $v = v_m$, $U(f, v)$ can be approximated by a parabola. Hence, Eq. (3) yields at $T \to 0$ that $\langle v \rangle_{\rm st}$ coincides with $v_m$.
[35] Despite $p_{\rm st}(v)$ does not exist in this case, one can define an infinite invariant density [36–39]. It can be useful to find asymptotic behavior of nonstationary moments corresponding to time-dependent $p(v, t)$ [36–39]. Developing pertinent analytical theory may become an important avenue in future research.
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for a small $f > 0$, apart from a minimum at some $v_m^{(1)}$ near $v = 0$, there exists a maximum of $U(v, f)$ at some $v_m^{(2)} > v_m^{(1)}$. This behavior is regularized by adding a small $\eta_0 \ll \eta_a$ to $\eta(v)$ in Eq. (7). Some viscoelastic materials show a stress overshoot [30], which can be described within such a model.

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