Coefficient of restitution of colliding viscoelastic spheres

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We perform a dimension analysis for colliding viscoelastic spheres to show that the coefficient of normal restitution ϵ depends on the impact velocity g as $\epsilon = 1-\gamma_1 g^{1/5}+\gamma_2 g^{2/5} \mp \cdots$, in accordance with recent findings. We develop a simple theory to find explicit expressions for coefficients γ_1 and γ_2. Using these and few next expansion coefficients for $\epsilon(g)$ we construct a Padé approximation for this function which may be used for a wide range of impact velocities where the concept of the viscoelastic collision is valid. The obtained expression reproduces quite accurately the existing experimental dependence $\epsilon(g)$ for ice particles. [S1063-651X(99)10410-0]

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I. INTRODUCTION

The change of relative velocity of inelastically colliding particles can be characterized by the coefficient of restitution ϵ. The normal component of the relative velocity after a collision $g' = \vec{v}_{12}'' \cdot \vec{e}$ follows from that before the collision $g = \vec{v}_{12} \cdot \vec{e}$ via

$$g' = -\epsilon g, \tag{1}$$

where $\vec{v}_1, \vec{v}_2$ and $\vec{v}_1', \vec{v}_2'$ are, respectively, the velocities before and after the collision, while the unit vector $\vec{e} \equiv \vec{r}_{12}/|\vec{r}_{12}|$ gives the direction of the inter-particle vector $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ at the instant of the collision.

From experiments as well as from theory it is well known that the coefficient of normal restitution ϵ is not a constant but it depends sensitively on the impact velocity [1–11]. Although most of the results in the field of granular gases have been derived neglecting this dependence but using a velocity-independent coefficient of restitution (e.g., [12–18]), it has been shown that the impact-velocity dependence of the coefficient of restitution has serious consequences for various problems in granular gas dynamics [19–24].

The equation of motion for inelastically colliding three-dimensional (3D) spheres has been addressed in [24–26], where the Hertz contact law [27]

$$F_{\rm el} = \rho \xi^{3/2}, \quad \rho \equiv \frac{2Y}{3(1-\nu^2)}\sqrt{R^{\rm eff}}, \tag{2}$$

for the elastic inter-particle force, has been extended to account for the viscoelasticity of the material which causes the dissipative part of the force

$$F_{\rm diss} = \frac{3}{2} A \rho \sqrt{\xi}\dot{\xi}. \tag{3}$$

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Here, ξ is the compression of the particles during the collision $\xi = R_1 + R_2 - |\vec{r}_1 - \vec{r}_2|$ (R_1, R_2 and $\vec{r}_1, \vec{r}_2$ are the radii and the positions of the spheres), Y and ν are, respectively, the Young modulus and the Poisson ratio of the particle material, $R^{\rm eff} \equiv R_1 R_2/(R_1+R_2)$, and the dissipative parameter A reads [25,26]

$$A = \frac{1}{3}\frac{(3\eta_2 - \eta_1)^2}{(3\eta_2 + 2\eta_1)}\left[\frac{(1-\nu^2)(1-2\nu)}{Y\nu^2}\right]. \tag{4}$$

The viscous constants η_1, η_2 relate the dissipative stress tensor to the deformation rate tensor [25,26,28]. The same functional dependence of $F_{\rm diss}(\xi,\dot{\xi})$ has been obtained in [29–31] using a different approach. We want to point out that Eqs. (3) and (4) do only hold if viscoelasticity is the only dissipative process during the particle collision. For the cases where plastic deformation, brittle failure, fracture, adhesion etc. have to be considered, there are more appropriate models for the particle contact, e.g., [32].

The equation of motion for inelastically colliding spheres reads, therefore,

$$\ddot{\xi} + \frac{\rho}{m^{\rm eff}}\left(\xi^{3/2} + \frac{3}{2}A\sqrt{\xi}\dot{\xi}\right) = 0, \tag{5}$$

with

$$\xi(0) = 0, \quad \dot{\xi}(0) = g,$$

and with $m^{\rm eff} \equiv m_1 m_2/(m_1+m_2)$($m_1, m_2$ are the masses of the colliding particles). To obtain the dependence of the restitution coefficient on the impact velocity for 3D spheres, the equation of motion (5) was solved numerically [10,24–26] and analytically [33], where the velocity-dependent restitution coefficient has been obtained as a series in powers of $g^{1/5}$:

$$\epsilon = 1 - C_1\left(\frac{3}{2}A\right)\left(\frac{\rho}{m^{\rm eff}}\right)^{2/5} g^{1/5} + C_2\left(\frac{3}{2}A\right)^2\left(\frac{\rho}{m^{\rm eff}}\right)^{4/5} g^{2/5} \mp \cdots . \tag{6}$$

The first coefficients $C_1 = 1.153\,44$ and $C_2 = 0.798\,26$ were evaluated analytically and then confirmed by numerical simulations [33].

Although in [33] a general method of derivation of *all* coefficients of the expansion (6) has been proposed, to obtain these, extensive calculations have to be performed. This approach does not provide closed-form expressions for the coefficients, but rather gives them in terms of convergent series which are to be evaluated up to the desired precision.

In the present study we show that a dimension analysis allows one to obtain the functional form of the $\epsilon(g)$ dependence for the elastic and dissipative forces. Within the framework of this analysis we reproduce the dependence (6) up to numerical values of coefficients C_k. A similar approach has been used by Tanaka [34] to prove that the constant coefficient of restitution is not consistent with physical reality (see also [10,35]). We also develop a simple approximative theory, which gives a continuum fraction representation for $\epsilon(g)$ and a closed-form expression for C_1 and C_2 with the same numerical values as above. Using then coefficients $C_1, \ldots, C_4$ (with C_3 and C_4 evaluated in the Appendix in accordance with the general scheme of Ref. [33]), we construct a Padé approximation, which reproduces fairly well the experimental data for colliding ice particles [5].

II. DIMENSIONAL ANALYSIS

To perform the general dimensional analysis we adopt the following form for the elastic and dissipative forces:

$$F_{\rm el} = m^{\rm eff} D_1 \xi^{\alpha},$$

$$F_{\rm diss} = m^{\rm eff} D_2 \xi^{\gamma} \dot{\xi}^{\beta}.$$

This general form (at least for small ξ and $\dot{\xi}$) follows from the fact that both elastic and dissipative forces vanish at $\xi = 0$ and $\dot{\xi} = 0$, respectively. With these notations the equation of motion for colliding particles reads

$$\ddot{\xi} + D_1 \xi^{\alpha} + D_2 \xi^{\gamma} \dot{\xi}^{\beta} = 0, \tag{7}$$

with

$$\xi(0) = 0, \quad \dot{\xi}(0) = g,$$

where g has already been introduced. Now we choose as the characteristic length ξ_0 of the problem, the maximal compression for the elastic case. It may be found from the condition that the initial kinetic energy $m^{\rm eff} g^2/2$ [36] equals the maximal elastic energy $m^{\rm eff} D_1 \xi_0^{\alpha+1}/(\alpha+1)$, which yields

$$\xi_0 = \left(\frac{\alpha+1}{2D_1}\right)^{1/(1+\alpha)} g^{2/(1+\alpha)}. \tag{8}$$

Choosing then the characteristic time of the problem as $\tau_0 = \xi_0/g$, we construct new dimensionless variables

$$\hat{\xi} = \xi/\xi_0, \quad \dot{\hat{\xi}} = \dot{\xi}/g, \quad \ddot{\hat{\xi}} = \frac{\xi_0}{g^2}\ddot{\xi}, \tag{9}$$

and recast the equation of motion into dimensionless form:

$$\ddot{\hat{\xi}} + \delta(g)\hat{\xi}^{\gamma}\dot{\hat{\xi}}^{\beta} + \frac{1+\alpha}{2}\hat{\xi}^{\alpha} = 0 \tag{10}$$

with

$$\hat{\xi}(0) = 0, \quad \dot{\hat{\xi}}(0) = 1,$$

$$\hat{\xi}(\tau_c) = 0, \quad \dot{\hat{\xi}}(\tau_c) = -\epsilon.$$

In the last equation (10) we supplemented the precollisional initial conditions at $\tau = 0$ with the after-collisional conditions at $\tau = \tau_c$ (τ is the dimensionless time and τ_c is the dimensionless duration of the collision). These follow just from the definition of the restitution coefficient. We point out that all dependence on the initial impact velocity on any quantity of the problem, including ϵ (this is just the dimensionless after-collisional velocity) comes only through the constant δ, which reads

$$\delta(g) = D_2\left(\frac{1+\alpha}{2D_1}\right)^{(1+\gamma)/(1+\alpha)} g^{2(\gamma-\alpha)/(1+\alpha)+\beta}. \tag{11}$$

Hence, $\epsilon(g) = \epsilon(\delta(g))$. A similar result for $\epsilon \to 0$, $\beta = 1$, and $\alpha = 3/2$ has been obtained in [37].

If we assume that the restitution coefficient does not depend on the impact velocity g, then it follows that

$$2(\gamma - \alpha) + \beta(1+\alpha) = 0. \tag{12}$$

For a linear dependence of the dissipative force on the velocity, i.e., for $\beta = 1$ (this seems to be the most realistic for small $\dot{\xi}$), one obtains a constant restitution coefficient for the following.

(i) the linear elastic force, $F_{\rm el} \sim \xi$, i.e. $\alpha = 1$. The condition (12) implies $\gamma = 0$, and thus the linear dissipative force $F_{\rm diss} \sim \dot{\xi}$.

(ii) the Hertz law for 3D spheres (2) $\alpha = 3/2$, therefore, $\gamma = \frac{1}{4}$ and $F_{\rm diss} \sim \dot{\xi}\xi^{1/4}$ provides a constant restitution coefficient.

We now ask the question: What kind of $\epsilon(g)$ dependence corresponds to the forces which act during collisions of viscoelastic particles? It may be generally shown [25,26,38] that the relation

$$F_{\rm diss} = A\dot{\xi}\frac{\partial}{\partial \xi}F_{\rm el}(\xi) \tag{13}$$

between the dissipative and elastic forces with the dissipative constant A given in Eq. (4) holds, provided the following three conditions are met [39].

(i) The elastic part of the stress tensor depends linearly on the deformation tensor [28].

(ii) The dissipative part of the stress tensor depends linearly on the deformation rate tensor [28].

(iii) The conditions of quasistatic motion are provided, i.e., $g \ll c$, $\tau_{\rm vis} \ll \tau_c$ [25,26] (here c is the speed of sound in the material of particles, $\tau_{\rm vis}$ is relaxation time of viscous processes in its bulk).

From this follows that $\beta=1$, $\gamma=\alpha-1$, and thus the constant restitution coefficient may be observed only for collisions of cubic particles with surfaces normal to the direction of collision. We wish to emphasize that this conclusion comes from the general analysis of viscoelastic collisions.

Let us discuss now collisions between spheres with elastic and dissipative forces as given by Eqs. (2) and (3), respectively. For these we have $m^{\rm eff}D_1=\rho$, $\alpha=3/2$, and $m^{\rm eff}D_2=\frac{3}{2}A\rho$, $\gamma=1/2$, and $\beta=1$ which yields the functional dependence for $\delta(g)$ and $\epsilon(g)$, respectively:

$$\delta=\frac{3}{2}\left(\frac{5}{4}\right)^{3/5}A\left(\frac{\rho}{m^{\rm eff}}\right)^{2/5}g^{1/5}, \tag{14}$$

$$\epsilon=\epsilon\left(A\left(\frac{\rho}{m^{\rm eff}}\right)^{2/5}g^{1/5}\right) \tag{15}$$

[skipping the prefactor of $\delta(g)$ in the last equation] in accordance with Eq. (6) as found previously.

III. RESTITUTION COEFFICIENT FOR SPHERES

Using $d/dt=\dot{\hat{\xi}}(d/d\hat{\xi})$ it is convenient to write the equation of motion for a collision in the form

$$\frac{d}{d\hat{\xi}}\left(\frac{1}{2}\dot{\hat{\xi}}^2+\frac{1}{2}\hat{\xi}^{5/2}\right)=-\delta\dot{\hat{\xi}}\hat{\xi}^{1/2}=\frac{dE(\hat{\xi})}{d\hat{\xi}},$$

$$\hat{\xi}(0)=0;\quad \dot{\hat{\xi}}(0)=1, \tag{16}$$

where we introduce the mechanical energy

$$E\equiv\frac{1}{2}\dot{\hat{\xi}}^2+\frac{1}{2}\hat{\xi}^{5/2}. \tag{17}$$

To find the energy losses in the first stage of the collision, which starts with zero compression and ends in the turning point with maximal compression $\hat{\xi}_0$,

$$\int_0^{\hat{\xi}_0}\frac{dE}{d\hat{\xi}}d\hat{\xi}=-\delta\int_0^{\hat{\xi}_0}\dot{\hat{\xi}}\hat{\xi}^{1/2}d\hat{\xi}, \tag{18}$$

one needs to know the dependence of the compression rate $\dot{\hat{\xi}}$ as a function of the compression $\hat{\xi}$.

For the case of elastic collisions, the maximal compression is $\hat{\xi}_0=1$, according to the definition of our dimensionless variables. Hence, the dependence $\dot{\hat{\xi}}(\hat{\xi})$ follows from the conservation of energy:

$$\dot{\hat{\xi}}(\hat{\xi})=\sqrt{1-\hat{\xi}^{5/2}}. \tag{19}$$

The velocity $\dot{\hat{\xi}}$ vanishes at the turning point $\hat{\xi}=1$. For inelastic collisions the maximal compression $\hat{\xi}_0$ is smaller than 1, therefore, one can write an approximation relation for the inelastic case:

$$\dot{\hat{\xi}}(\hat{\xi})\approx\sqrt{1-(\hat{\xi}/\hat{\xi}_0)^{5/2}}, \tag{20}$$

which also gives vanishing velocity $\dot{\hat{\xi}}$ at the turning point $\hat{\xi}_0$. Integration in Eq. (18) may be performed yielding

$$\frac{1}{2}\hat{\xi}_0^{5/2}-\frac{1}{2}=-\delta\, d\,\hat{\xi}_0^{3/2}, \tag{21}$$

where we take into account that $E(\hat{\xi}_0)=\frac{1}{2}\hat{\xi}_0^{5/2}$, $E(0)=\frac{1}{2}\dot{\hat{\xi}}(0)=\frac{1}{2}$, and introduce a constant

$$d\equiv\int_0^1 x^{1/2}\sqrt{1-x^{5/2}}=\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{5}\right)}{5\Gamma\left(\frac{21}{10}\right)}. \tag{22}$$

Consider now the inverse collision, which is defined as a collision which starts with velocity ϵg and ends with velocity g. According to the concept of the inverse collision introduced in [33] (which is a useful auxiliary model), it is characterized by a negative damping (the energy is ''pumped'' into the system during the collision). The maximal compression $\hat{\xi}_0$ is the same in both collisions, the direct and the inverse.

Rescaling equation of motion for the inverse collision in the very same way as for the direct collision yields

$$\frac{dE(\hat{\xi})}{d\hat{\xi}}=+\delta\dot{\hat{\xi}}\hat{\xi}^{1/2},$$

$$\hat{\xi}(0)=0,\quad \dot{\hat{\xi}}(0)=\epsilon. \tag{23}$$

This suggests the following approximative relation for $\dot{\hat{\xi}}(\hat{\xi})$ during the inverse collision:

$$\dot{\hat{\xi}}(\hat{\xi})\approx\epsilon\sqrt{1-(\hat{\xi}/\hat{\xi}_0)^{5/2}}, \tag{24}$$

with the additional prefactor ϵ, which is the initial velocity in the inverse collision.

Integration of the energy *gain* for the first stage of the inverse collision (which equals up to its sign the energy loss in the second stage of the direct collision [33]) may be performed in just the same way as for the direct collision, yielding the result

$$\frac{1}{2}\hat{\xi}_0^{5/2}-\frac{\epsilon^2}{2}=+\epsilon\delta\, d\,\hat{\xi}_0^{3/2}, \tag{25}$$

where we again use $E(\hat{\xi}_0)=\frac{1}{2}\hat{\xi}_0^{5/2}$ and $E(0)=\frac{1}{2}\epsilon^2$. Multiplying Eq. (21) by ϵ and summing it up with Eq. (25) we obtain a simple approximative relation between the restitution coefficient and the (dimensionless) maximal compression:

TABLE I. Coefficients of the Padé formula (35) as derived from the coefficients a_k.

$d_0=a_4-2a_3-a_2^2+3a_2-1$	
$d_1=[1-a_2+a_3-2a_4+(a_2-1)(3a_2-2a_3)]d_0^{-1}$	$=2.5839$
$d_2=[(a_3-a_2)(1-2a_2)-a_4]d_0^{-1}$	$=3.5839$
$d_3=[a_3+a_2^2(a_2-1)-a_4(a_2+1)]d_0^{-1}$	$=2.9839$
$d_4=[a_4(a_3-1)+(a_3-a_2)(a_2^2-2a_3)]d_0^{-1}$	$=1.1487$
$d_5=[2(a_3-a_2)(a_4-a_2a_3)-(a_4-a_2^2)^2-a_3(a_3-a_2^2)]d_0^{-1}$	$=0.3265$

$$\epsilon=\hat{\xi}_0^{5/2}. \tag{26}$$

Substituting this into Eq. (21) we arrive at an equation for the restitution coefficient

$$\epsilon+2\,\delta\,d\,\epsilon^{3/5}=1. \tag{27}$$

The formal solution to this equation may be written as a continuum fraction (which does not diverge in the limit $g\to\infty$):

$$\epsilon^{-1}=1+2\,\delta\,d(1+2\,\delta\,d(1+\cdots)^{2/5}\cdots)^{2/5}. \tag{28}$$

Another way of representing the restitution coefficient ϵ is a series expansion in terms of δ. For practical applications it is convenient to return to dimensional units. We define the characteristic velocity g^* such that

$$\delta\equiv\frac{1}{2d}\left(\frac{g}{g^*}\right)^{1/5}, \tag{29}$$

with d being defined in Eq. (22). Using the definition (14) together with Eq. (22) we find for the characteristic velocity

$$(g^*)^{-1/5}=\frac{\sqrt{\pi}}{2^{1/5}5^{2/5}}\,\frac{\Gamma(3/5)}{\Gamma(21/10)}\left(\frac{3}{2}A\right)\left(\frac{\rho}{m^{\mathrm{eff}}}\right)^{2/5}. \tag{30}$$

Evaluating the numerical prefactor finally yields

$$(g^*)^{-1/5}=1.153\,44\left(\frac{3}{2}A\right)\left(\frac{\rho}{m^{\mathrm{eff}}}\right)^{2/5}. \tag{31}$$

Note that the numerical constant 1.153 44 has to be equal to C_1 in Eq. (6).

With this new notation the restitution coefficient reads

$$\epsilon=1-a_1\left(\frac{g}{g^*}\right)^{1/5}+a_2\left(\frac{g}{g^*}\right)^{2/5}-a_3\left(\frac{g}{g^*}\right)^{3/5}+a_4\left(\frac{g}{g^*}\right)^{4/5}+\cdots, \tag{32}$$

with $a_1=1$, $a_2=3/5$ (which are exact values), $a_3=6/25=0.24$, $a_4=7/125=0.056,\ldots$ (which deviate from the correct ones; see below). Comparing Eq. (32) with Eq. (6), we conclude that our simple approximative theory reproduces exactly the coefficients C_1 and C_2, which were found before using extensive analysis [33].

We also performed rigorous but elaborated calculations according to the general scheme of [33] to find the exact coefficients (details are given in the Appendix)

$$a_3=0.315\,119,\quad a_4=0.161\,167, \tag{33}$$

or, respectively,

$$C_3=-0.483\,582,\quad C_4=0.285\,279. \tag{34}$$

Hence, we observe that while the first two coefficients $a_1=1$ and $a_2=3/5$ are correctly obtained from the approximative theory, the next approximated coefficients a_3,a_4 differ from the exact ones.

For practical applications, such as molecular dynamics simulations, however, the expansion (32) is of limited value, due to its divergence for high impact velocities, $g\to\infty$. According to the velocity distribution function there is a certain probability that the relative velocity g of colliding particles exceeds the limit of applicability of Eq. (32). Therefore, we use the obtained coefficients to construct a Padé approximation for $\epsilon(g)$, which reveals the correct limits of the boundary conditions, $\epsilon(0)=1$ and $\epsilon(\infty)=0$. Since the dependence $\epsilon(g)$ is expected to be a smooth, monotonically decreasing function, we choose a ''1-4'' Padé approximation:

$$\epsilon=\frac{1+d_1\left(\frac{g}{g^*}\right)^{1/5}}{1+d_2\left(\frac{g}{g^*}\right)^{1/5}+d_3\left(\frac{g}{g^*}\right)^{2/5}+d_4\left(\frac{g}{g^*}\right)^{3/5}+d_5\left(\frac{g}{g^*}\right)^{4/5}}. \tag{35}$$

Standard analysis yields the coefficients d_k in terms of the coefficients a_k [40] (see Table I).

Using the characteristic velocity $g^*=0.32$ cm/s for ice as a fitting parameter we could reproduce fairly well the experimental dependence of the restitution coefficient of ice as a function of the impact velocity g in the whole range of g (Fig. 1). The discrepancy with the experimental data at small g follows from the fact that the extrapolation expression, $\epsilon=0.32/g^{0.234}$, used in [5] has an unphysical divergence at $g\to0$ and does not imply the fail of the theory for this region. The scattering of the experimental data presented in [5] is large for small impact velocity according to experimental complications, hence the fit formula of [5] cannot be expected to be accurate enough for too small velocities. Moreover, in the region of very small velocity, it is possible that something other than viscoelastic interactions might influence the collision behavior, e.g., adhesion. Similarly, for very high velocities, effects such as brittle failure, fracture, and others may contribute to dissipation.

IV. CONCLUSION

We developed a dimensional analysis for the inelastic collision of spherical particles. We could show that for 3D spheres the functional form for $\epsilon(g)$ agrees with that derived

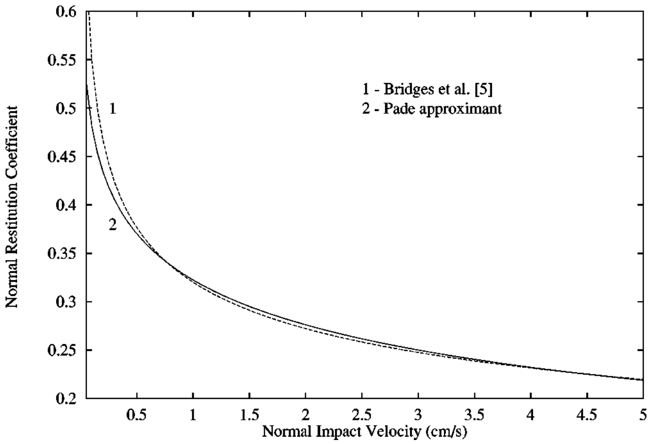

FIG. 1. Dependence of the normal restitution coefficient on the impact velocity for ice particles. Solid line, experimental data of [5]; dashed line, the Padé approximation (35) with the constants given in the table and with the characteristic velocity for ice $g^* = 0.32$ cm/s.

previously [33], using a much more complicated approach. Using a simple approximative theory we found a continuum-fraction representation for $\epsilon(g)$ and obtained explicit expressions for the coefficients of the series expansion of the restitution coefficient in terms of the impact velocity. The first two coefficients in this series coincide with that found previously by numerical evaluation. Next, we also report on a few coefficients which we have derived within the general approach of a previous study [33]. Using the first four coefficients of this series expansion we constructed a Padé approximation for $\epsilon(g)$. It reproduces fairly well the experimental data for colliding ice particles.

The restitution coefficient as a function of the impact velocity contains the parameter g^* which depends on the elastic and viscous material properties [see Eqs. (30),(2), and (4)]. If these properties are known, g^* is univocally determined. Otherwise, g^* can be determined experimentally; if the restitution coefficient ϵ is known for one specified impact velocity, one can determine g^* and the full function $\epsilon(g)$ is known. The obtained relation for the restitution coefficient may be used for a wide range of the impact velocities, provided that the energy loss during a collision is attributed to viscoelasticity and that all the other dissipative processes (plastic deformation, fragmentation of particles) may be ignored.

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APPENDIX

The general method of derivation of the expansion coefficients C_k has been given in [33]. Here we briefly sketch the main lines of derivation and provide some details for the particular cases of C_3 and C_4. Since the method of derivation is based on the collection of terms with different dependence on the initial velocity g, it is convenient to use a scaling, somewhat different from that used before for the dimensional analysis. Namely, we rescale the time as $t' = (\rho/m_{\rm eff})^{2/5} g^{1/5} t$ and the length as $x = (\rho/m_{\rm eff})^{2/5}\xi$ to recast Eq. (5) into the form [41]

$$x'' + \alpha g^{-1/5} x' \sqrt{x} + g^{-2/5} x^{3/2} = 0, \tag{A1}$$

with $\alpha \equiv \frac{3}{2} A(\rho/m_{\rm eff})^{2/5}$, and using all the notations introduced previously. The initial conditions for the rescaled Eq. (A1) now read $x(0)=0$ and $x'(0)=g^{4/5}$. For simplicity of notations we will keep, in what follows, t for the rescaled time. As it was shown in [33], the trajectory may be expanded in terms of $\sqrt{t}$ as

$$x(t) = b_1 t^{1/2} + b_2 t + b_3 t^{3/2} + b_4 t^2 + b_5 t^{5/2} + b_6 t^3 + b_7 t^{7/2} + \cdots . \tag{A2}$$

Clearly, both b_1 and b_3 should be zero to avoid divergence of velocity and acceleration at $t=0$. At the same time $b_2 = g^{4/5}$ and $b_4 = 0$, due to the equation of motion at vanishing compression. This yields

$$x(t) = g^{4/5} t + b_5 t^{5/2} + b_6 t^3 + b_7 t^{7/2} + \cdots . \tag{A3}$$

From Eq. (A3) one obtains $x'(t)$ and $x''(t)$ which are to be substituted into the equation of motion (A1). One also needs $\sqrt{x}$ and $x^{3/2}$; the expansions for these in terms of $\sqrt{t}$ read

$$\sqrt{x} = g^{2/5} t^{1/2} + \frac{b_5}{2g^{2/5}} t^2 + \frac{b_6}{2g^{2/5}} t^{5/2} + \frac{b_7}{2g^{2/5}} t^3 + \cdots \tag{A4}$$

and

$$x^{3/2} = g^{6/5} t^{3/2} + \frac{3}{2} g^{2/5} b_5 t^3 + \frac{3}{2} g^{2/5} b_6 t^{7/2} + \cdots . \tag{A5}$$

Inserting the expansions for $x'(t)$, $x''(t)$, $\sqrt{x}$, and $x^{3/2}$ into Eq. (A1), and collecting the orders of t, we obtain

$$0 = \left(\frac{15}{4} b_5 + \alpha g^{1/5}\right) t^{1/2} + 6 b_6 t + \left(\frac{35}{4} b_7 + 1\right) t^{3/2} + (12 b_8 + 3\alpha g^{1/5} b_5) t^2 + \left(\frac{63}{4} b_9 + \frac{7}{2} \alpha g^{1/5} b_6\right) t^{5/2}. \tag{A6}$$

This suggests the coefficients

$$b_5 = -\frac{4}{15} \alpha g^{1/5}, \tag{A7}$$

$$b_6 = 0, \tag{A8}$$

$$b_7 = -\frac{4}{35}, \tag{A9}$$

$$b_8 = \frac{1}{15} \alpha^2 g^{2/5}, \tag{A10}$$

$$b_9=0, \tag{A11}$$

so that the solution for the trajectory finally reads

$$x(t)=g^{4/5}t-\frac{4}{15}\alpha g t^{5/2}-\frac{4}{35}g^{4/5}t^{7/2}+\frac{1}{15}\alpha^2 g^{6/5}t^4+\cdots. \tag{A12}$$

In order to get the higher orders, which is conceptionally simple but requires extensive calculus, we wrote a program [42], that by formula manipulations, performs exactly the steps we described above and which is able to find the trajectory up to any desired order.

Generally, it is convenient to write the solution as a series:

$$x(t)=g^{4/5}(x_0(t)+\alpha g^{1/5}x_1(t)+\alpha^2 g^{2/5}x_2(t)+\cdots). \tag{A13}$$

Here $x_0(t)$ is a ''zero-order'' trajectory, which refers to the case of undamped collision, the ''first-order'' trajectory, $x_1(t)$, accounts for damping in linear (with respect to α) approximation, the ''second-order'' trajectory, $x_2(t)$, corresponds to the next approximation $\sim\alpha^2$, etc. Here we give our results for these ''n-order'' trajectories up to $n=3$, obtained using the above mentioned program up to the order t^{11}:

$$x_0=t-\frac{4}{35}t^{7/2}+\frac{1}{175}t^6-\frac{22}{104\,125}t^{17/2}+\frac{52}{8\,017\,625}t^{11},$$

$$x_1=-\frac{4}{15}t^{5/2}+\frac{3}{70}t^5-\frac{713}{238\,875}t^{15/2}+\frac{61\,216}{42\,639\,187}t^{10},$$

$$x_2=\frac{1}{15}t^4-\frac{937}{75\,075}t^{13/2}+\frac{871}{808\,500}t^9, \tag{A14}$$

$$x_3=-\frac{38}{2475}t^{11/2}+\frac{43\,943}{13\,513\,500}t^8-\frac{1\,184\,627}{3\,594\,591\,000}t^{21/2}.$$

To proceed we need to find the maximal compression $x_{\max}$, which is reached at time $t_{\max}$. The point of maximal compression is a turning point of the trajectory, where the velocity is zero. Therefore, the condition

$$x'_{\max}(t_{\max})=0 \tag{A15}$$

holds at this point. With the above expression for the trajectory [Eqs. (A13) and (A14)], the last equation (A15) is an equation to determine $t_{\max}$, which may be then used to find $x_{\max}$. This equation, however, is a high-order algebraic equation for $\sqrt{t_{\max}}$, which is not generally solvable. On the other hand, for the undamped collision, $t_{\max}$ equals one half of the collision duration t_c^0 and both quantities of interest are known [28]:

$$t_{\max}^0=\frac{t_c^0}{2}=\left(\frac{4}{5}\right)^{3/5}\frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{9}{10}\right)}, \tag{A16}$$

$$x_0\left(\frac{t_c^0}{2}\right)=\left(\frac{5}{4}\right)^{2/5}.$$

For a viscoelastic collision $t_{\max}$ certainly differs from $t_c^0/2$, so that $t_{\max}=t_c^0/2+\delta t$. If the dissipation parameter α is not large, the deviation δt is presumably small; therefore, we expand $x'(t_{\max})=x'(t_c^0/2+\delta t)$ in terms of δt:

$$\begin{aligned}g^{-4/5}x'(t_{\max})=&\left[x_0'\left(\frac{t_c^0}{2}\right)+\delta t x_0''\left(\frac{t_c^0}{2}\right)+\frac{\delta t^2}{2}x_0'''\left(\frac{t_c^0}{2}\right)+\cdots\right]\\&+\alpha g^{1/5}\left[x_1'\left(\frac{t_c^0}{2}\right)+\delta t x_1''\left(\frac{t_c^0}{2}\right)\right.\\&\left.+\frac{\delta t^2}{2}x_1'''\left(\frac{t_c^0}{2}\right)+\cdots\right]\\&+\alpha^2 g^{2/5}\left[x_2'\left(\frac{t_c^0}{2}\right)+\delta t x_2''\left(\frac{t_c^0}{2}\right)+\cdots\right]\\&+\alpha^3 g^{3/5}\left[x_3'\left(\frac{t_c^0}{2}\right)+\cdots\right]+\cdots=0,\end{aligned} \tag{A17}$$

where we use representation (A13) for the trajectory. The deviation δt vanishes at $\alpha=0$ and, thus, suggests the expansion in terms of α:

$$\delta t=\tau_1\alpha+\tau_2\alpha^2+\tau_3\alpha^3\cdots. \tag{A18}$$

Substituting δt, given by Eq. (A18), into Eq. (A17) and collecting terms of the same order of α yields

$$Y_0+\alpha Y_1+\alpha^2 Y_2+\alpha^3 Y_3+\cdots=0, \tag{A19}$$

with the abbreviations

$$Y_0=x_0'\left(\frac{t_c^0}{2}\right),$$

$$Y_1=\tau_1 x_0''\left(\frac{t_c^0}{2}\right)+g^{1/5}x_1'\left(\frac{t_c^0}{2}\right),$$

$$Y_2=\tau_2 x_0''\left(\frac{t_c^0}{2}\right)+\frac{\tau_1^2}{2}x_0'''\left(\frac{t_c^0}{2}\right)+g^{1/5}\tau_1 x_1''\left(\frac{t_c^0}{2}\right)+g^{2/5}x_2'\left(\frac{t_c^0}{2}\right), \tag{A20}$$

$$\begin{aligned}Y_3=&\tau_3 x_0''\left(\frac{t_c^0}{2}\right)+\tau_1\tau_2 x_0'''\left(\frac{t_c^0}{2}\right)+\frac{\tau_1^3}{6}x'''{}'_0\left(\frac{t_c^0}{2}\right)+g^{1/5}\tau_2 x_1''\left(\frac{t_c^0}{2}\right)\\&+g^{1/5}\frac{\tau_1^2}{2}x_1'''\left(\frac{t_c^0}{2}\right)+g^{2/5}\tau_1 x_2''\left(\frac{t_c^{0\prime}}{2}\right)+g^{3/5}x_3'\left(\frac{t_c^{0\prime}}{2}\right).\end{aligned}$$

The conditions $Y_k=0$ for $k=0,\ldots,3$, together with Eq. (A20), allows us to express the constants τ_1, τ_2, τ_3, etc. in terms of functions $x_1(t)$, $x_2(t)$, $x_3(t)$, etc., and their time derivatives taken at time $(t_c^0/2)$:

$$\tau_1 = -g^{1/5}\frac{x_1'\left(\frac{t_c^0}{2}\right)}{x_0''\left(\frac{t_c^0}{2}\right)}, \tag{A21}$$

$$\tau_2 = g^{2/5}\left[-\frac{x_1'^2\left(\frac{t_c^0}{2}\right)x_0'''\left(\frac{t_c^0}{2}\right)}{2x_0''^3\left(\frac{t_c^0}{2}\right)} + \frac{x_1'\left(\frac{t_c^0}{2}\right)x_1''\left(\frac{t_c^0}{2}\right)}{x_0''^2\left(\frac{t_c^0}{2}\right)} - \frac{x_2'\left(\frac{t_c^0}{2}\right)}{x_0''\left(\frac{t_c^0}{2}\right)}\right],$$

We do not write the expression for τ_3 since, due to the special properties of the problem, i.e., due to the fact that $x_0'(t_c^0/2)=0$, the value τ_3 is not needed for calculation of ϵ up to fourth order of α. The functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ are known and given by Eqs. (A14), so that the constants τ_1 and τ_2 may be found explicitly.

Writing the maximal compression as

$$x_{\max} = g^{4/5}\left[x_0\left(\frac{t_c^0}{2}+\delta t\right) + \alpha g^{1/5}x_1\left(\frac{t_c^0}{2}+\delta t\right) + \alpha g^{2/5}x_2\left(\frac{t_c^0}{2}+\delta t\right) + \alpha^3 g^{3/5}x_3\left(\frac{t_c^0}{2}+\delta t\right)\right], \tag{A22}$$

and expanding this in terms of δt, using then representation of δt as $\delta t = \alpha\tau_1 + \alpha^2\tau_2 + \cdots$, with τ_1, τ_2 from Eq. (A21) and collecting terms of the same order of α, we obtain

$$x_{\max} = g^{4/5}(y_0 + \alpha g^{1/5}y_1 + \alpha^2 g^{2/5}y_2 + \alpha^3 g^{3/5}y_3), \tag{A23}$$

where $y_0, \ldots, y_3$ are pure numbers:

$$y_0 = x_0\left(\frac{t_c^0}{2}\right) = 1.093\,362, \tag{A24}$$

$$y_1 = x_1\left(\frac{t_c^0}{2}\right) = -0.504\,455, \tag{A25}$$

$$y_2 = \left[x_2\left(\frac{t_c^0}{2}\right) - \frac{1}{2}\frac{x_1'^2\left(\frac{t_c^0}{2}\right)}{x_0''\left(\frac{t_c^0}{2}\right)}\right] = 0.260\,542, \tag{A26}$$

$$y_3 = \left[x_3\left(\frac{t_c^0}{2}\right) - \frac{x_1'\left(\frac{t_c^0}{2}\right)x_2'\left(\frac{t_c^0}{2}\right)}{x_0''\left(\frac{t_c^0}{2}\right)} + \frac{1}{2}\frac{x_1'^2\left(\frac{t_c^0}{2}\right)x_1''\left(\frac{t_c^0}{2}\right)}{x_0''\left(\frac{t_c^0}{2}\right)} - \frac{x_1'^3\left(\frac{t_c^0}{2}\right)x_0'''}{x_0''^3(t_c^0/2)}\right] = -0.136\,769, \tag{A27}$$

and where we use expressions (A14) for $x_1(t)$, $x_2(t)$, and $x_3(t)$.

To calculate the coefficient of restitution, one has to use the concept of inverse collision, as was introduced in [33] and discussed in previous chapters of the present study. One obtains the solution of this inverse collision by replacing $g \to \epsilon g$ for the initial velocity and $\alpha \to -\alpha$ for the dissipative coefficient. In particular, this applies to the maximal compression of the inverse collision $x_{\max}^{\mathrm{inv}} = x_{\max}(g \to \epsilon g,\ \alpha \to -\alpha)$. For consistency one has to require the maximum compressions for direct and inverse collision to be equal, i.e.,

$$x_{\max}^{\mathrm{inv}} = x_{\max}, \tag{A28}$$

or using Eq. (A23),

$$\epsilon^{4/5}g^{4/5}(y_0 - \alpha\epsilon^{1/5}g^{1/5}y_1 + \alpha^2\epsilon^{2/5}g^{2/5}y_2 - \alpha^3\epsilon^{3/5}g^{3/5}y_3 + \cdots) = g^{4/5}(y_0 + \alpha g^{1/5}y_1 + \alpha^2 g^{2/5}y_2 + \alpha^3 g^{3/5}y_3 + \cdots). \tag{A29}$$

Equation (A29) is, in fact, an algebraic equation for $\epsilon^{1/5}$, which may not be generally solved. For this reason we write ϵ as an expansion of $\alpha g^{1/5}$, which is the only combination in which both parameters appear:

$$\epsilon = 1 + C_1\alpha g^{1/5} + C_2(\alpha g^{1/5})^2 + C_3(\alpha g^{1/5})^3 + C_4(\alpha g^{1/5})^4 + \ldots, \tag{A30}$$

and substitute Eq. (A30) into Eq. (A29). Collecting orders we find

$$\left[-\frac{4}{5}y_0C_1 + 2y_1\right]\alpha g^{1/5} + \left[\left(-\frac{4}{5}C_2 + \frac{2}{25}C_1^2\right)y_0 + y_1C_1\right]\times\alpha^2 g^{2/5} + \left[\left(-\frac{4}{5}C_3 + \frac{4}{25}C_1C_2 - \frac{4}{125}C_1^3\right)y_0 + y_1C_2 - \frac{6}{5}y_2C_1 + 2y_3\right]\alpha^3 g^{3/5} + \left\{\left[-\frac{4}{5}C_4 + \frac{2}{25}(C_2^2 + 2C_1C_3) - \frac{12}{125}C_1^2C_2 + \frac{11}{625}C_1^4\right]\times y_0 + y_1C_3 + \left(-\frac{6}{5}C_2 - \frac{3}{25}C_1^2\right)y_2 + \frac{7}{5}y_3C_1\right\}\times\alpha^4 g^{4/5} = 0. \tag{A31}$$

The last equation (A31) yields the final result for the coefficients:

$$C_1 = \frac{5}{2}\frac{y_1}{y_2} = -1.153\,449, \tag{A32}$$

$$C_2 = \frac{15}{4}\left(\frac{y_1}{y_2}\right)^2 = \frac{3}{5}C_1^2 = 0.798\,267,$$

$$C_3 = \frac{95}{16}\left(\frac{y_1}{y_2}\right)^3 - \frac{15}{4}\frac{y_2y_1}{y_0^2} + \frac{5}{2}\frac{y_3}{y_0} = -0.483\,582,$$

$$C_4 = \frac{315}{32}\left(\frac{y_1}{y_2}\right)^4 - \frac{105}{8}\frac{y_2y_1^2}{y_0^3} + \frac{35}{4}\frac{y_3y_1}{y_0^2} = 0.285\,279.$$

Using $(g^*)^{-1/5}=C_1\alpha$, we obtain for coefficients a_k in expansion (32):

$$a_1=1, \tag{A33}$$

$$a_2=C_2/C_1^2=3/5, \tag{A34}$$

$$a_3=C_3/C_1^3=0.315\,119 \tag{A35}$$

$$a_4=C_4/C_1^4=0.161\,167. \tag{A36}$$

Note that although the general method given in this appendix allows one to evaluate up to a desired precision *all*, in principle, coefficients C_k, it does not provide the closed-form expression for C_1 as the simple approximate approach given in the main text does.

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[38] Derivation of the dissipative force given in [25,26] for colliding spheres may be straightforwardly generalized to obtain the relation (13) [or Eq. (A17) in [25,26]] for colliding bodies of any shape, provided that displacement field in the bulk of the material of bodies in contact is a one-valued function of the compression (see also [39]).

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[40] Obviously, the coefficients of the Padé approximation may be chosen up to an arbitrary factor to multiply numerator and denominator; we chose it to have unity as a leading term for both of these.

[41] Note that, in difference to the calculations in the main part of the article the quantities x, x', and x'' do have units, namely $(\mathrm{m/sec})^{4/5}$. The rescaled time is dimensionless. The purpose of this scaling was only to simplify the dependence of the problem on the material parameters, it was necessary to keep the explicit dependence of the problem on the initial velocity.

[42] The maple-program is available at URL: http://summa.physik.hu-berlin.de/~kies/papers/DimAnalysis/epsilon_simple.html