Granular Gases – The Early Stage

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Abstract. We investigate the evolution of the velocity distribution function of a granular gas composed of viscoelastic particles in the homogeneous cooling state, i.e. before clustering occurs. The deviation of the velocity distribution function from the Maxwellian distribution is quantified by a Sonine polynomials expansion. The first non-vanishing Sonine coefficient $a_2(t)$, reveals a complex time dependence which allows to assign the granular gas the property of an age. We discuss the possibility to measure the age of a granular gas.

1   Introduction

Granular Gases as rarefied systems of granular particles in the absence of gravity may be exemplified by a cloud of interstellar dust. Similar as molecular gases Granular Gases may be described within the concepts of classical Statistical Mechanics, such as temperature T, velocity distribution function $f(v)$, etc. Once initialized with a certain velocity distribution, Granular Gases cool down due to inelastic collisions of their particles. Although these systems are extremely simple, in principle, they reveal a variety of structure formation and much work has been done recently to characterize the properties of cooling Granular Gases (see [1] with many references therein).

Most of these results have been obtained under the assumption that the coefficient of restitution ϵ which characterizes the loss of energy of two colliding particles (see below), is a material constant. The assumption $\epsilon = $ const., however, does not only contradict experiments which show that ϵ depends significantly on the impact velocity [2,3], but it contradicts even some basic mechanical laws [4]. The simplest physically correct description of dissipative particle collisions is based on the assumption of viscoelastic material deformation during collisions [5], which is valid for particle collisions in a certain range of impact velocity and is in good agreement with experimental data [3,6].

We investigate the statistical properties of Granular Gases of viscoelastic particles for which the dependence of the restitution coefficient on the impact velocity $\epsilon = \epsilon(v_{\mathrm{imp}})$ is known [4,7]. Starting from a homogeneous distribution we study the early stage of its evolution, where no spatial structures, as clusters [8,9] and vortexes [10], have emerged yet. This stage is called the homogeneously cooling state. Our results show that the properties of Granular Gases change qualitatively if one takes into account viscoelastic material properties,

i.e. $\epsilon = \epsilon(v_{\mathrm{imp}})$ [11], as compared with the equivalent system, but under the oversimplified assumption $\epsilon = \mathrm{const.}$.

In the next Section we discuss briefly the impact-velocity dependence of the normal restitution coefficient. In Sec. 3 we introduce the method to study the velocity distribution by means of the Boltzmann equation with the Sonine polynomials expansion formalism, and discuss the results obtained for gases of particles interacting with a constant restitution coefficient. Our main results, which describe the time evolution of the velocity distribution for granular gases of viscoelastic particles are derived in Sec. 4. In Sec. 5 we discuss the concept of the age of a granular gas which is based on the time evolution of the velocity distribution. Finally, in conclusion we summarize our findings.

2 Two-Particle Interaction of Viscoelastic Spheres

The microscopic dynamics of granular particles is governed by the (normal) restitution coefficient ϵ which relates the normal components of the particle velocities before and after a collision, $\boldsymbol{v}_{ij} \equiv \boldsymbol{v}_i - \boldsymbol{v}_j$ and $\boldsymbol{v}_{ij}^* \equiv \boldsymbol{v}_i^* - \boldsymbol{v}_j^*$ by $\left|\boldsymbol{v}_{ij}^*\boldsymbol{e}\right| = \epsilon\left|\boldsymbol{v}_{ij}\boldsymbol{e}\right|$. The unit vector $\boldsymbol{e} = \boldsymbol{r}_{ij}/\left|\boldsymbol{r}_{ij}\right|$ gives the direction of the inter-center vector $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ at the instant of the collision. From the conservation of momentum one finds the change of velocity for the colliding particles:

$$\boldsymbol{v}_i^* = \boldsymbol{v}_i - \frac{1}{2}\left(1+\epsilon\right)\left(\boldsymbol{v}_{ij}\cdot\boldsymbol{e}\right)\boldsymbol{e}\,, \qquad \boldsymbol{v}_j^* = \boldsymbol{v}_j + \frac{1}{2}\left(1+\epsilon\right)\left(\boldsymbol{v}_{ij}\cdot\boldsymbol{e}\right)\boldsymbol{e}\,. \qquad (1)$$

For elastic collisions one has $\epsilon = 1$ and for inelastic collisions ϵ decreases with increasing degree of inelasticity.

In literature it is frequently assumed that the restitution coefficient is a material constant, $\epsilon = \mathrm{const.}$ Experiments, e.g. [2,3], as well as theoretical investigations [5] show, however, that this assumption is not consistent with the nature of inelastic collisions, it does not agree even with a dimension analysis [4]. The impact velocity dependence of the restitution coefficient $\epsilon\left(v_{\mathrm{imp}}\right) = \epsilon\left(\left|\boldsymbol{e}\,\boldsymbol{v}_{ij}\right|\right)$ has been obtained by generalizing Hertz's contact problem to viscoelastic spheres [5]. From the generalized Hertz equation [12] one obtains the velocity-dependent restitution coefficient for viscoelastic spheres [7]

$$\epsilon = 1 - C_1 A\alpha^{2/5}\left|\boldsymbol{e}\,\boldsymbol{v}_{ij}\right|^{1/5} + C_2 A^2\alpha^{4/5}\left|\boldsymbol{e}\,\boldsymbol{v}_{ij}\right|^{2/5} \mp \cdots \qquad (2)$$

with

$$\alpha = \left(\frac{3}{2}\right)^{3/2}\frac{Y\sqrt{R^{\mathrm{eff}}}}{m^{\mathrm{eff}}\left(1-\nu^2\right)}\,, \qquad (3)$$

where Y is the Young modulus, ν is the Poisson ratio, and A depends on dissipative parameters of the particle material (for details see [5]). The effective mass and radius are defined as

$$R^{\mathrm{eff}} = R_1 R_2/(R_1 + R_2) \qquad m^{\mathrm{eff}} = m_1 m_2/(m_1 + m_2) \qquad (4)$$

with $R_{1/2}$ and $m_{1/2}$ being the radii and masses of the colliding particles. The constants are given by [4,7]

$$C_1 = \frac{\Gamma(3/5)\sqrt{\pi}}{2^{1/5}5^{2/5}\Gamma(21/10)} \approx 1.1534 \qquad C_2 = \frac{3}{5}C_1^2 \approx 0.7982 \,. \tag{5}$$

Equation (2) refers to the case of pure viscoelastic interaction, i.e. when the relative velocity $|\boldsymbol{v}_{ij}\boldsymbol{e}|$ is not too large (to avoid plastic deformation of the particles) and is not too small (to allow to neglect surface effects such as roughness, adhesion and van der Waals interactions). The dependence of $\epsilon = \epsilon(|\boldsymbol{ev}_{ij}|)$ (without the material dependence) was already mentioned in [6] where heuristic arguments have been applied.

In what follows we consider a granular gas of identical viscoelastic spheres of unit mass.

3   Kinetics of Granular Gases: The Case $\epsilon = $ const.

The evolution of the velocity distribution function is generally described by the Boltzmann-Enskog equation, which for the force-free case reads [13,14,15]:

$$\frac{\partial}{\partial t}f(\boldsymbol{v},t) = g_2(\sigma)\sigma^2 \int d\boldsymbol{v}_2 \int d\boldsymbol{e}\,\Theta(-\boldsymbol{v}_{12}\cdot\boldsymbol{e})\,|\boldsymbol{v}_{12}\cdot\boldsymbol{e}|$$
$$\times\,\{\chi f(\boldsymbol{v}_1^{**},t)f(\boldsymbol{v}_2^{**},t) - f(\boldsymbol{v}_1,t)f(\boldsymbol{v}_2,t)\} \equiv g_2(\sigma)I(f,f)\,, \tag{6}$$

where $\sigma = 2R$ is the diameter of particles, $g_2(\sigma) = (2-\eta)/2(1-\eta)^3$ ($\eta = \frac{1}{6}\pi n\sigma^3$ is packing fraction) denotes the contact value of the two-particle correlation function [16], which accounts for the increasing collision frequency due to the excluded volume effects; $\Theta(x)$ is the Heaviside step-function. The velocities $\boldsymbol{v}_1^{**}$ and $\boldsymbol{v}_2^{**}$ refer to the precollisional velocities of the so-called inverse collision, which results with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as the after-collisional velocities. Finally the factor χ in the gain term appears respectively from the Jacobian of the transformation $d\boldsymbol{v}_1^{**}d\boldsymbol{v}_2^{**} \to d\boldsymbol{v}_1 d\boldsymbol{v}_2$ and from the relation between the lengths of the collisional cylinders $\epsilon|\boldsymbol{v}_{12}^{**}\cdot\boldsymbol{e}|\,dt = |\boldsymbol{v}_{12}\cdot\boldsymbol{e}|\,dt$. [17]. For the constant restitution coefficient $\epsilon = $ const. this reads $\chi = \epsilon^{-2}$ [14,18], i.e., it is independent on the impact velocity and, therefore, independent on time.

With the scaling Ansatz for the distribution function

$$f(\boldsymbol{v},t) = \frac{n}{v_0^3(t)}\tilde{f}\left(\frac{v}{v_0(t)}\right) = \frac{n}{v_0^3(t)}\tilde{f}(\boldsymbol{c})\,, \tag{7}$$

where n is the number density of the granular gas and $v_0(t)$ is the thermal velocity defined by

$$\frac{3}{2}nT(t) = \int d\boldsymbol{v}\frac{v^2}{2}f(\boldsymbol{v},t)\,,= \frac{3}{2}nv_0^2(t) \tag{8}$$

the Boltzmann equation may be reduced to two independent equations: one for the (time-independent) scaling function $\tilde{f}(\boldsymbol{c})$, and the other one for the time-dependence of the thermal velocity (i.e. for the temperature). Solving the equation for the temperature one obtains $T(t) = T_0/\left(1 + t/\tau\right)^2$ [13].

The solution of the other equation for $\tilde{f}(\boldsymbol{c})$, may be found in terms of the Sonine polynomial expansion [14,18,19]. For the case of $\epsilon = $ const. it reads

$$\tilde{f}(\boldsymbol{c}) = \phi(c)\left\{1 + \sum_{p=1}^{\infty} a_p S_p(c^2)\right\} , \tag{9}$$

where $\phi(c) \equiv \pi^{-d/2} \exp(-c^2)$ is the Maxwellian distribution for the rescaled velocity, and the first few Sonine polynomials are

$$S_0(x) = 1, \quad S_1(x) = -x^2 + \frac{3}{2}, \quad S_2(x) = \frac{x^2}{2} - \frac{5x}{2} + \frac{15}{8}. \tag{10}$$

The leading (zero-order) term in Eq. (9) is the Maxwellian distribution, while the next-order terms, characterized by the coefficients a_i describe the deviation of the distribution from the Maxwellian. For the case of $\epsilon = $ const. the velocity distribution function $\tilde{f}(\boldsymbol{c})$ is time-independent, therefore, the coefficients of the Sonine polynomials expansion are constants.

If the inelasticity is small, one can restrict to the first non-vanishing term beyond the Maxwellian which has the coefficient a_2 ($a_1 = 0$, according to the definition of temperature [14,18,19]). For $\epsilon = $ const. this coefficient reads [20]

$$a_2 = \frac{16(1 - \epsilon)(1 - 2\epsilon^2)}{9 + 24d + 8\epsilon d + 41\epsilon + 30(1 - \epsilon)\epsilon^2} . \tag{11}$$

A more accurate expression for a_2 may be found in [19].

4   Kinetics of Granular Gases: Viscoelastic Particles

For viscoelastic particles the restitution coefficient ϵ depends on the impact velocity due to Eq. (2). Hence, the factor χ in the Boltzmann equation (6) is not anymore constant as for $\epsilon = $ const. but it reads

$$\chi = 1 + \frac{11}{5}C_1 A \alpha^{2/5} \left|\boldsymbol{v}_{12} \cdot \boldsymbol{e}\right|^{1/5} + \frac{66}{25}C_1^2 A^2 \alpha^{4/5} \left|\boldsymbol{v}_{12} \cdot \boldsymbol{e}\right|^{2/5} + \cdots \tag{12}$$

Using again the scaling Ansatz (7) the rhs of the Boltzmann equation does not factorize into two parts, one depending only on the scaling function $\tilde{f}(\boldsymbol{c})$, and another one, depending only on v_0. Consequently, one can not obtain a set of decoupled equations for $\tilde{f}(\boldsymbol{c})$ and for the temperature.

Nevertheless, it is worth to substitute the generalized scaling Ansatz

$$f(\boldsymbol{v}, t) = \frac{n}{v_0^3(t)}\tilde{f}(\boldsymbol{c}, t) \tag{13}$$

into the kinetic equation (6). After some algebra Eq.(6) may be recast into the form

$$\frac{\mu_2}{3}\left(3 + c_1\frac{\partial}{\partial c_1}\right)\tilde{f}(\boldsymbol{c}, t) + B^{-1}\frac{\partial}{\partial t}\tilde{f}(\boldsymbol{c}, t) = \tilde{I}\left(\tilde{f}, \tilde{f}\right) \tag{14}$$

where we define the dimensionless collisional integral

$$\tilde{I}\left(\tilde{f}, \tilde{f}\right) = \int d\boldsymbol{c}_2 \int d\boldsymbol{e}\,\Theta(-\boldsymbol{c}_{12}\cdot\boldsymbol{e})|\boldsymbol{c}_{12}\cdot\boldsymbol{e|}\left\{\tilde{\chi}\tilde{f}(\boldsymbol{c}_1^{**}, t)\tilde{f}(\boldsymbol{c}_2^{**}, t) - \tilde{f}(\boldsymbol{c}_1, t)\tilde{f}(\boldsymbol{c}_2, t)\right\} \tag{15}$$

with the reduced factor $\tilde{\chi}$

$$\tilde{\chi} = 1 + \frac{11}{5}C_1\delta'\,|\boldsymbol{c}_{12}\cdot\boldsymbol{e}|^{1/5} + \frac{66}{25}C_1^2\delta'^{\,2}\,|\boldsymbol{c}_{12}\cdot\boldsymbol{e}|^{2/5} + \cdots \tag{16}$$

which depends now on time via a quantity

$$\delta'(t) \equiv A\alpha^{2/5}\,[2T(t)]^{1/10} \equiv \delta\,[2T(t)/T_0]^{1/10} \tag{17}$$

Here $\delta \equiv A\alpha^{2/5}[T_0]^{1/10}$ and T_0 is the initial temperature. We also define $B = B(t) \equiv v_0(t)g_2(\sigma)\sigma^2 n$, and the moments of the dimensionless collision integral

$$\mu_p \equiv -\int d\boldsymbol{c}_1 c_1^p \tilde{I}\left(\tilde{f}, \tilde{f}\right) . \tag{18}$$

According to the definitions (8), (18) and of B, the second moment μ_2 defines the evolution of the temperature:

$$\frac{dT}{dt} = \frac{1}{3}g_2(\sigma)\sigma^2 n v_0^3 \int d\boldsymbol{c}_1 c_1^2 \tilde{I}\left(\tilde{f}, \tilde{f}\right) = -\frac{2}{3}BT\mu_2 . \tag{19}$$

The velocity distribution function we again describe by a Sonine polynomials expansion as introduced in Eq. (9). Since in contrast to the case $\epsilon = \text{const.}$, the Boltzmann equation for a gas of viscoelastic particles does not factorize into a time dependent equation for temperature and a time-independent equation for the velocity distribution, the Sonine coefficients a_i are not constants but depend explicitly on time, i.e. one has now

$$\tilde{f}(\boldsymbol{c}, t) = \phi(c)\left\{1 + \sum_{p=1}^{\infty} a_p(t)S_p(c^2)\right\} . \tag{20}$$

Equations for $a_p(t)$ may be found by multiplying both sides of Eq.(14) with c_1^p and integrating over $d\boldsymbol{c}_1$. One obtains

$$\frac{\mu_2}{3}p\langle c^p\rangle - B^{-1}\sum_{k=1}^{\infty}\dot{a}_k\nu_{kp} = \mu_p \tag{21}$$

where integration by parts has been performed and where we define

$$\nu_{kp} \equiv \int \phi(c)c^p S_k(c^2)dc; \quad \langle c^p\rangle \equiv \int c^p \tilde{f}(\boldsymbol{c}, t)d\boldsymbol{c}. \tag{22}$$

From (21) we see already that the granular temperature and the Sonine coefficients and, hence, the distribution function do not evolve independently.

The calculation of ν_{kp} is straightforward; the first few of them read $\nu_{22} = 0$, $\nu_{24} = \frac{15}{4}$. The odd moments $\langle c^{2n+1} \rangle$ are zero, while the even ones, $\langle c^{2n} \rangle$ may be expressed in terms of a_k with $0 \le k \le n$. Calculations show that $\langle c^2 \rangle = \frac{3}{2}$, implying $a_1 = 0$, according to the definition of temperature (8) (e.g.[14]), and that $\langle c^4 \rangle = \frac{15}{4} (1 + a_2)$.

Now we assume that the dissipation is not large so that the deviation from the Maxwellian distribution may be sufficiently described by the second term in the expansion (9) only, with all higher-order terms with $p > 2$ discarded. Then (21) is an equation for the coefficient a_2. Using the above results for ν_{22}, ν_{24}, $\langle c^2 \rangle$ and $\langle c^4 \rangle$ it is easy to show that Eq.(21) converts for $p = 2$ into identity, while for $p = 4$ it reads:

$$\dot{a}_2 - \frac{4}{3} B\mu_2 (1 + a_2) + \frac{4}{15} B\mu_4 = 0 . \tag{23}$$

With the approximation $\tilde{f} = \phi(c)[1 + a_2(t)S_2(c^2)]$ the time-dependent coefficients $\mu_p(t)$ may be expressed in terms of a_2 due to the definition (18). Using the properties of the collision integral (e.g. the conservation of the total momentum at collision) one can obtain relations for the $\mu_p(t)$ (e.g. [14]):

$$\mu_p = -\frac{1}{2} \int d\boldsymbol{c}_1 \int d\boldsymbol{c}_2 \int d\boldsymbol{e} \Theta(-\boldsymbol{c}_{12} \cdot \boldsymbol{e}) |\boldsymbol{c}_{12} \cdot \boldsymbol{e}| \phi(c_1)\phi(c_2) \times$$
$$\left\{ 1 + a_2 \left[S_2(c_1^2) + S_2(c_2^2) \right] + a_2^2 S_2(c_1^2)S_2(c_2^2) \right\} \Delta(c_1^p + c_2^p)$$

where $\Delta\psi(\boldsymbol{c}_i) \equiv [\psi(\boldsymbol{c}_i^*) - \psi(\boldsymbol{c}_i)]$ denotes the change of some function $\psi(\boldsymbol{c}_i)$ in a direct collision. Calculations up to the second order in δ yield [17]

$$\mu_2 = \sum_{k=0}^{2} \sum_{n=0}^{2} \mathcal{A}_{kn} \delta'^k a_2^n \tag{24}$$

where the coefficients $\mathcal{A}$ are pure numbers: $\mathcal{A}_{00} = 0$, $\mathcal{A}_{01} = 0$, $\mathcal{A}_{02} = 0$, $\mathcal{A}_{10} = \omega_0$, $\mathcal{A}_{11} = \frac{6}{25}\omega_0$, $\mathcal{A}_{12} = \frac{21}{2500}\omega_0$, $\mathcal{A}_{20} = \omega_1$, $\mathcal{A}_{21} = \frac{119}{400}\omega_1$ and $\mathcal{A}_{22} = \frac{4641}{640000}\omega_1$, with $\omega_0 \equiv 2\sqrt{2\pi}2^{1/10}\Gamma\left(\frac{21}{10}\right)C_1 \approx 6.48562$ and $\omega_1 \equiv \sqrt{2\pi}2^{1/5}\Gamma\left(\frac{16}{5}\right)C_1^2 \approx 9.28569$. Similarly

$$\mu_4 = \sum_{k=0}^{2} \sum_{n=0}^{2} \mathcal{B}_{kn} \delta'^k a_2^n \tag{25}$$

with $\mathcal{B}_{00} = 0$, $\mathcal{B}_{01} = 4\sqrt{2\pi}$, $\mathcal{B}_{02} = \frac{1}{8}\sqrt{2\pi}$, $\mathcal{B}_{10} = \frac{56}{10}\omega_0$, $\mathcal{B}_{11} = \frac{1806}{250}\omega_0$, $\mathcal{B}_{12} = \frac{567}{12500}\omega_0$, $\mathcal{B}_{20} = \frac{77}{10}\omega_1$, $\mathcal{B}_{21} = \frac{149054}{13750}\omega_1$ and $\mathcal{B}_{22} = \frac{348424}{5500000}\omega_1$.

Thus, Eqs.(23) and (19), together with Eqs.(24) and (25) form a closed set to find the time evolution of the temperature and of the coefficient a_2. In contrast to the case of constant restitution coefficient where $a_2 = $ const., in a gas of viscoelastic particles the time evolution of temperature is coupled to the time

evolution of the Sonine coefficient a_2. This coupling may lead to a rather peculiar time-evolution of the system.

Introducing the reduced temperature $u(t) \equiv T(t)/T_0$ we recast the set (23), (19) into the form

$$\dot{u} + \frac{1}{\tau_0} u^{\frac{8}{5}} \left(\frac{5}{3} + \frac{2}{5} a_2 + \frac{7}{500} a_2^2 \right) - \frac{1}{\tau_0} q_1 \delta\, u^{\frac{17}{10}} \left(\frac{5}{3} + \frac{119}{240} a_2 + \frac{1547}{128000} a_2^2 \right) = 0 \quad (26)$$

$$\dot{a}_2 - r_0 u^{1/2} \mu_2 \left(1 + a_2 \right) + \frac{1}{5} r_0 u^{1/2} \mu_4 = 0 \quad (27)$$

where we introduce the characteristic time

$$\tau_0^{-1} = \frac{16}{5} q_0 \delta \cdot \tau_c(0)^{-1} = \frac{16}{5} q_0 \delta \left(4\sqrt{\pi} g_2(\sigma) \sigma^2 n \sqrt{T_0} \right)^{-1} \quad (28)$$

and define $q_0 = 2^{1/5} \Gamma(21/10) C_1 / 8 \approx 0.173318$, $q_1 \equiv 2^{1/10}(\omega_1/\omega_0) \approx 1.53445$, and $r_0^{-1} \equiv (24\sqrt{2\pi}/5) q_0 \delta \tau_0$. Equation (28), as shown below, describes the time evolution of the temperature. To obtain the last equations we use the expressions for μ_2, μ_4, B, and for the coefficients $\mathcal{A}$. Note that the characteristic time τ_0 is $\delta^{-1} \gg 1$ times larger than the mean collision time $\tau_c(0)$.

We will find the solution to these equations as expansions in terms of the small dissipative parameter δ ($\delta'(t) = \delta \cdot 2^{1/10} u^{1/10}(t)$):

$$u = u_0 + \delta \cdot u_1 + \delta^2 \cdot u_2 + \cdots, \qquad a_2 = a_{20} + \delta \cdot a_{21} + \delta^2 \cdot a_{22} + \cdots \quad (29)$$

Substituting Eqs. (29), (24) and (25) into Eqs. (26), (27), one can solve these equations perturbatively for each order of δ.

Keeping only linear terms with respect to δ, one can find the analytical solution (see [17] for details). This reads e.g. for the coefficient $a_2(t)$:

$$a_2(t) = \delta \cdot a_{21} = -\frac{12}{5} w(t)^{-1} \left\{ \mathrm{Li}\left[w(t)\right] - \mathrm{Li}\left[w(0)\right] \right\} \quad (30)$$

where $w(t) \equiv \exp\left[(q_0\delta)^{-1} (1 + t/\tau_0)^{1/6} \right]$ and $\mathrm{Li}(x)$ is the logarithmic integral. For $t \ll \tau_0$ Eq.(30) reduces to

$$a_2(t) = -\delta \cdot h \left(1 - e^{-8t/15\tau_c(0)} \right) \quad (31)$$

where $h \equiv 2^{1/10} \left(\mathcal{B}_{10} - 5\mathcal{A}_{10} \right) / 16\pi = 0.415964$. As it follows from Eq.(31), after a transient time of the order of few collisions per particle, i.e. for $\tau_c(0) < t \ll \tau_0$, $a_2(t)$ saturates at the "steady-state" (on the time-scale $\sim \tau_c(0)$) value $-h\,\delta = -0.415964\delta$. For $t \gg \tau_0$ it decays on a "slow" time-scale $\sim \tau_0$:

$$a_2(t) \simeq -\delta \cdot h \left(t/\tau_0 \right)^{-1/6} \quad (32)$$

and the velocity distribution tends to the Maxwellian. Linear theory gives for the temperature for $t \gg \tau_0$ [17]

$$\frac{T(t)}{T_0} = \left(1 + \frac{t}{\tau_0} \right)^{-5/3} + \left(\frac{12}{25} h + 2q_1 \right) \left(\frac{t}{\tau_0} \right)^{-11/6} \quad (33)$$

with the constants h and q_1 given above. The linear theory agrees fairly well with numerical solution (see Fig. 1 where the time dependence of a_2 is given) for small δ.
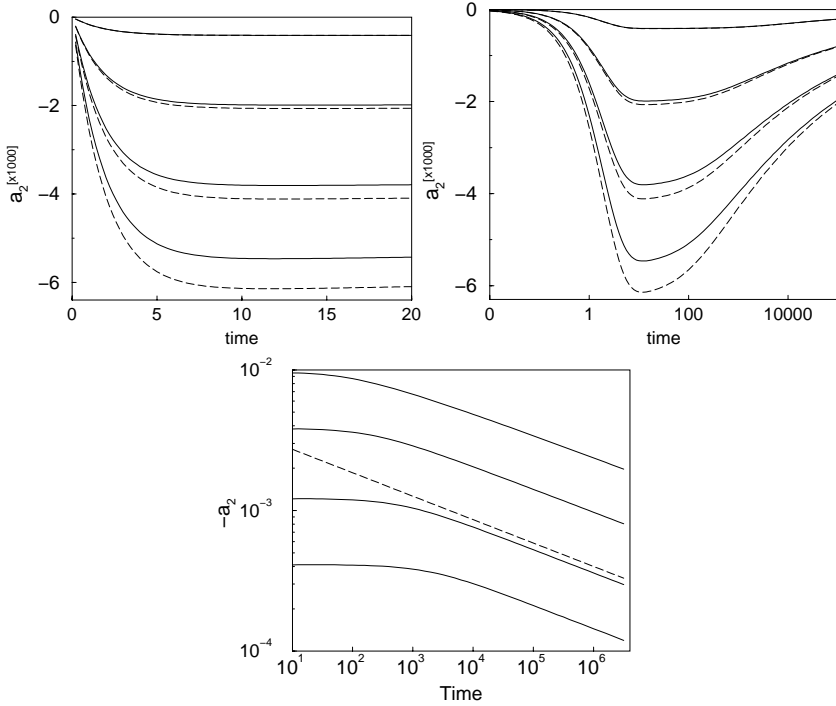

Fig. 1. Time dependence of the second coefficient of the Sonine polynomial expansion $a_2(t) \times 100$. Time is given in units of the mean collision time $\tau_c(0)$. $\delta = 0.1, 0.11, 0.12, \ldots, 0.20$ (bottom to top).

For larger values of δ the linear theory breaks down and we performed only numerical study of the equations. The results are given in Fig.2. As compared to the case of small δ, an additional intermediate regime in the time-evolution of the velocity distribution is observed. The first "fast" stage of evolution takes place, as before, on the time scale of few collisions per particle, where maximal deviation from the Maxwellian distribution is achieved (Fig.2). For $\delta \geq 0.15$ these maximal values of a_2 are positive. Then, on the second stage (intermediate regime), which continues $10 - 100$ collisions, a_2 changes its sign and reaches a maximal negative deviation. Finally, on the third, slow stage, $a_2(t)$ relaxes to zero on the slow time-scale $\sim \tau_0$, just as for small δ. In Fig.2 (left) we show the first stage of the time evolution of $a_2(t)$ for systems with large δ. At a certain value of the dissipative parameter δ the behavior changes qualitatively, i.e. the system then reveals another time scale as discussed above.
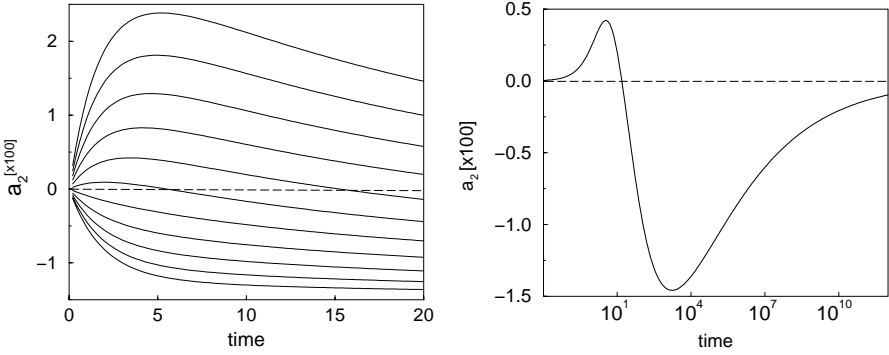
Fig. 2. The Sonine coefficient $a_2(t)$ for larger dissipation δ (numerical results). Time is given in units of mean collisional time $\tau_c(0)$. Left: $a_2(t) \times 100$ for $\delta = 0.1, 0.11, 0.12, \ldots, 0.20$ (bottom to top). Right: The plot of $a_2(t) \times 100$ for $\delta = 0.16$ over logarithmic time shows all stages of evolution discussed in the text.

5   The Age of a Granular Gas

Assume that the distribution function at time of initialization of a granular gas is known to be Maxwellian, e.g. from the nature of the physical process which gave rise to the granular gas. Then the calculation in the previous section describes the evolution of the velocity distribution function quantified by the first non-trivial Sonine coefficient a_2. We have seen that this quantity evolves characteristically in time. Assume we know an experimental method to measure the velocity distribution over a certain time interval (t_1, t_2) and, therefore, to trace the time evolution of a_2 in this interval. The time dependence of a_2 is known theoretically (at least numerically) as the solution of the set of differential equations (26,27), which depends on the parameters τ_0 and δ, i.e. $a_2 = a_2(t, \tau_0, \delta)$ (τ_0 and δ, in their turn, depend on the material parameters ρ and A). This suggests a method to *measure* the age of a granular gas, i.e., to determine the time of its initialization.

Let us explain this in more detail. If we know $a_2(t)$, $t_1 \leq t \leq t_2$ we can compute the values of τ_0 and δ, which completely parameterize the dependence $a_2(t)$, with an accuracy, which depends on the size of the time interval $t_2 - t_1$. Hence, using the solution (either numerical or analytical) of the set of equations (26,27) we can trace back the dependence $a_2(t)$ for times $t < t_1$. The time t_0 when the curve $a_2(t)$ cuts the abscissa corresponds to the Maxwellian distribution, i.e., it gives the initialization time and, therefore, the age of the gas. In Fig. 2 (right) we see that for larger dissipation there are two such times when $a_2\left(t_0^{(1)}\right) = a_2\left(t_0^{(2)}\right) = 0$ ($0 = t_0^{(1)} < t_0^{(2)}$) but only the earlier one, $t_0^{(1)}$, corresponds to the age (see Fig. 3). Thus, we need a method to discriminate between them: If $a_2(t)$ was positive for at least a part of the time interval (t_1, t_2), then the initialization time t_0 is uniquely determined (Fig. 3a). If the value of a_2 was negative (Fig. 3b,c) we can trace time-backwards the dependence $a_2(t)$ until at time t_0 the condition

$a_2(t_0) = 0$ is fulfilled. Then additional analysis is required: The value of δ which was already computed determines the "steady-state" value of a_2 which is reached after the quick relaxation in a time of the order of few mean-collision times. For small dissipation δ it is negative and for large dissipation it is positive. Thus, if the steady state value of a_2 is negative, the time t_0 with $a_2(t_0) = 0$ corresponds to the initialization time, i.e. $t_0 = t_0^{(1)}$. Otherwise (i.e. for positive steady-state a_2), t_0 corresponds to $t_0^{(2)} > 0$, and one has to trace $a_2(t)$ further time-backward in order to find the next time t_0 which fulfills $a_2(t_0) = 0$ to find the time of initialization.
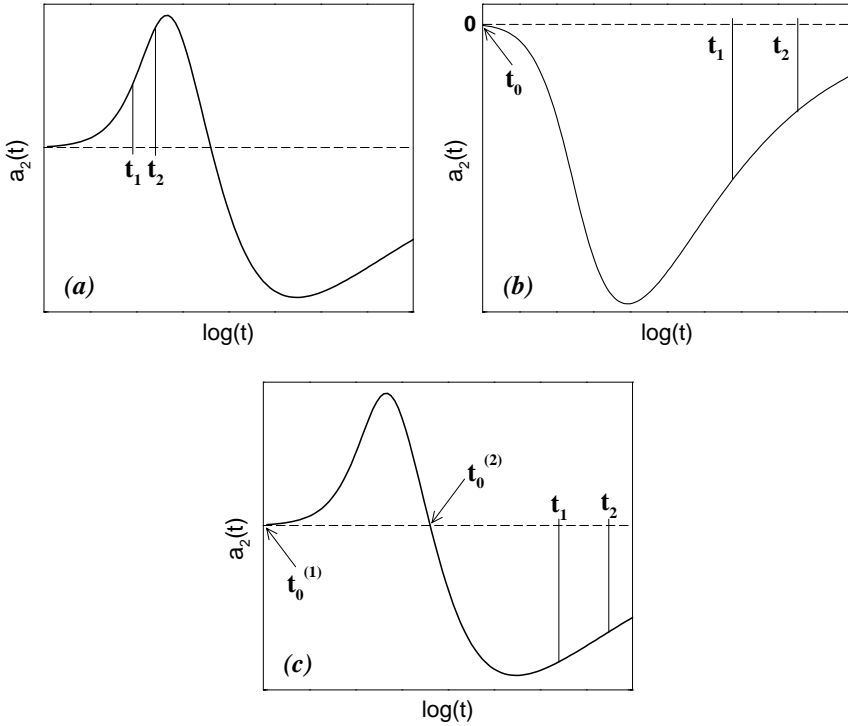

Fig. 3. Illustration of the method to compute the age of the granular gas: If the velocity distribution and, hence, $a_2(t)$ can be measured in the interval (t_1, t_2), the function $a_2(t)$ can be traced backwards in time due to the described theory. If $a_2(t) > 0$ for $t_1 < t < t_2$ (Figure (a)), the condition $a_2(t_0) = 0$ yields the age of the gas. If $a_2(t) < 0$ for $t_1 < t < t_2$ depending on material properties there may be one time $a_2(t_0) = 0$ (Figure (b)) or two times for which $a_2(t_0^{(1)}) = a_2(t_0^{(2)}) = 0$ (Figure (c)). To discriminate the cases (b) and (c) one needs further consideration (see text).

If the age of the gas has been measured according to the described method one can also calculate the initial temperature T_0, i.e. the initial energy of the gas. From δ and T_0 according to the definition of δ one can estimate the combination

of the material parameters $A\alpha^{2/5}$, and even the size of the particles σ using Eq.(28), which relates τ_0, T_0, δ and $\sigma^2 g_2(\sigma)$, provided the number density n may be measured.

Assume an astrophysical catastrophic impact took place at a certain time and produced a granular gas cloud with Maxwellian velocity distribution. If one would be able to measure the velocity distribution function in a much later time interval $t_1 \leq t \leq t_2$ following the described procedure one would be able to determine (i) the time when the impact took place, (ii) the energy of the impacting bodies (from the initial temperature), (iii) some material properties of the bodies and (iv) the grain size of the granular gas. The described analysis does not require the knowledge of the material properties of the particles.

6   Conclusion

We analyzed the time evolution of the velocity distribution function in a granular gas of viscoelastic particles in the homogeneous cooling state. The assumption of viscoelasticity is the simplest assumption for the dissipative collision of particles which is in agreement with mechanical laws. The collision of these particles is characterized by an impact-velocity dependent restitution coefficient.

For the case of a gas of particles which interact via a constant restitution coefficient its evolution is completely determined by the time dependence of the temperature. The velocity distribution function has a simple scaling form, i.e., it depends only on the reduced velocity of the particles, $\boldsymbol{c} = \boldsymbol{v}/v_0(t)$, which is just the velocity measured in units of the characteristic velocity $v_0(t)$, related to the current temperature. The scaling form, thus, persists with time.

Contrary, the velocity distribution function of a gas of viscoelastic particles does not have a simple scaling form. The deviation of the velocity distribution from the Maxwellian which is for the case $\epsilon = \text{const.}$ a function of ϵ only, i.e. time-independent, depends for a gas of viscoelastic particles explicitly on time, i.e. the velocity distribution function undergoes a time evolution. We quantify the deviation from the Maxwellian distribution by means of the first non-vanishing term of the Sonine polynomials expansion, characterized by the coefficient a_2. We assume that inelasticity of the particles is small and, hence, higher order terms may be neglected. Contrary to the case of the constant restitution coefficient, where $a_2 = \text{const.}$, for a gas of viscoelastic particles a_2 reveals a rather complex time behavior with different regimes of evolution.

The time dependence of the distribution function, quantified by $a_2(t)$, exhibits different stages of evolution, which allows to assign a Granular Gas the term "aging". The explicit time dependence implies that the process has a definite starting point of initialization, i.e. earlier times do not correspond to a physically meaningful state of the system. At first glance it might appear cumbersome that a granular gas, i.e. a cooling gas of dissipatively colliding particles, has a well defined initialization time. Indeed, if the temperature was the only characteristics of the system there would be no reason to mention some specific initial temperature, since the gas may exist at any temperature. In this case we

would not be able to conclude whether the present state of a gas is the starting point of its evolution or an intermediate one, so that its history started long ago at much higher temperature. This is the case if the particles interact via a constant restitution coefficient $\epsilon = $ const. For gases of viscoelastic particles, however, the velocity distribution evolves in a way which allows to determine univocally the time-lag from the starting point and, hence, the age of the granular gas.

From the knowledge of the velocity distribution in a certain time interval $t_1 \leq t \leq t_2$ we can determine not only the age of the gas but also its initial temperature and certain material properties of the particles. This result may be useful to determine the time, the energy and other system properties of catastrophic impacts in astrophysical systems.

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