

Granular Gases with Impact-Velocity-Dependent Restitution Coefficient

Nikolai V. Brilliantov^{1,2} and Thorsten Pöschel²

¹ Moscow State University, Physics Department, Moscow 119899, Russia.
e-mail: nbrillia@physik.hu-berlin.de

² Humboldt-Universität, Institut für Physik, Invalidenstr. 110, D-10115 Berlin,
Germany. e-mail: thorsten@physik.hu-berlin.de,
<http://summa.physik.hu-berlin.de/~thorsten>

Abstract. We consider collisional models for granular particles and analyze the conditions under which the restitution coefficient might be a constant. We show that these conditions are not consistent with known collision laws. From the generalization of the Hertz contact law for viscoelastic particles we obtain the coefficient of normal restitution ϵ as a function of the normal component of the impact velocity v_{imp} . Using $\epsilon(v_{\text{imp}})$ we describe the time evolution of temperature and of the velocity distribution function of a granular gas in the homogeneous cooling regime, where the particles collide according to the viscoelastic law. We show that for the studied systems the simple scaling hypothesis for the velocity distribution function is violated, i.e. that its evolution is not determined only by the time dependence of the thermal velocity. We observe, that the deviation from the Maxwellian distribution, which we quantify by the value of the second coefficient of the Sonine polynomial expansion of the velocity distribution function, does not depend on time monotonously. At first stage of the evolution it increases on the mean-collision time-scale up to a maximum value and then decays to zero at the second stage, on the time scale corresponding to the evolution of the granular gas temperature. For granular gas in the homogeneous cooling regime we also evaluate the time-dependent self-diffusion coefficient of granular particles. We analyze the time dependence of the mean-square displacement and discuss its impact on clustering. Finally, we discuss the problem of the relevant internal time for the systems of interest.

1 Introduction

Granular gases, i.e. systems of inelastically colliding particles, are widely spread in nature. They may be exemplified by industrial dust or interstellar dust; the behavior of matter in planetary rings is also described in terms of the granular gas dynamics. As compared with common molecular gases, the steady removal of kinetic energy in these systems due to dissipative collisions causes a variety of nonequilibrium phenomena, which have been very intensively studied (e.g. [1–11]). In most of these studies the coefficient of restitution, which characterizes the energy lost in the collisions, was

assumed to be constant. This approximation, although providing a considerable simplification, and allowing to understand the main effects in granular gas dynamics, is not always justified (see also the paper by Thornton in this book [12]). Moreover, sometimes it occurs to be too crude to describe even qualitatively the features of granular gases. Here we discuss the properties of granular gases consisting of viscoelastically colliding particles which implies an impact-velocity dependent restitution coefficient. The results are compared with results for gases consisting of particles which interact via a constant restitution coefficient and we see that the natural assumption of viscoelasticity leads to *qualitative* modifications of the gas properties.

The following problems will be addressed:

- Why does the restitution coefficient ϵ depend on the impact velocity v_{imp} ?
- How does it depend on the impact velocity?
- What are the consequences of the dependence of ϵ on v_{imp} on the collective behavior of particles in granular gases? In particular how does $\epsilon = \epsilon(v_{\text{imp}})$ influence:
 - the evolution of temperature with time?
 - the evolution of the velocity distribution function with time?
 - the self-diffusion in granular gases?

In what follows we will show that the dependence of the restitution coefficient on the impact velocity is a very basic property of dissipative particle collisions, whereas the assumption of a constant restitution coefficient for the collision of three-dimensional spheres may lead to a physically incorrect dependence of the dissipative force on the compression rate of the colliding particles. From the Hertz collision law and the general relation between the elastic and dissipative forces we deduce the dependence of the restitution coefficient on the impact velocity, which follows purely from scaling considerations. We also give the corresponding relation obtained from rigorous theory. Using the dependence $\epsilon(v_{\text{imp}})$ we derive the time dependence of the temperature, the time-evolution of the velocity distribution function and describe self-diffusion in granular gases in the homogeneous cooling regime.

2 Dependence of the Restitution Coefficient on the Impact Velocity

The collision of two particles may be characterized by the compression ξ and by the compression rate $\dot{\xi}$, as shown on Fig. 1. The compression gives rise to the elastic force $F_{\text{el}}(\xi)$, while the dissipative force $F_{\text{diss}}(\xi, \dot{\xi})$ appears due to the compression rate.

If the compression and the compression rate are not very large, one can assume the dependence of the elastic and dissipative force on ξ and $\dot{\xi}$

$$F_{\text{el}}(\xi) \sim \xi^\alpha \quad (1)$$

$$F_{\text{diss}}(\xi, \dot{\xi}) \sim \dot{\xi}^\beta \xi^\gamma. \quad (2)$$

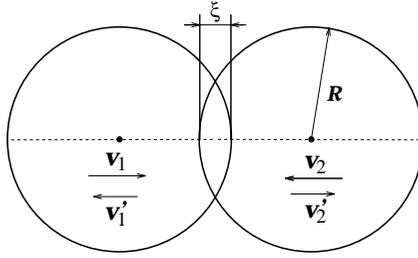


Fig. 1. Sketches of two colliding spheres. The compression ξ is equal to $2R - |\mathbf{r}_1 - \mathbf{r}_2|$, with $\mathbf{r}_{1/2}$ being the particles positions. The compression rate is $\dot{\xi} = v_1 - v_2$. For simplicity the head on collision of identical spheres is shown.

The dimension analysis yields the following functional form for the dependence of the restitution coefficient on the impact velocity [13]:

$$\epsilon(v_{\text{imp}}) = \epsilon \left(v_{\text{imp}}^{\frac{2(\gamma - \alpha)}{1 + \alpha} + \beta} \right) \tag{3}$$

Therefore, the condition for a constant restitution coefficient imposes the relation between the exponents α , β and γ [14, 15]:

$$2(\gamma - \alpha) + \beta(1 + \alpha) = 0. \tag{4}$$

For compressions which do not exceed the plasticity threshold, the particle’s material behaves as a viscoelastic medium. Then it may be generally shown [16–18] that the relation

$$F_{\text{diss}} = A \dot{\xi} \frac{\partial}{\partial \xi} F_{\text{el}}(\xi) \tag{5}$$

between the elastic and dissipative force holds, independently on the shape of the bodies in contact, provided three conditions are met [19]:

- (i) The elastic components of the stress tensor σ_{el}^{ik} depend linearly on the components of the deformation tensor u_{ik} [20].
- (ii) The dissipative components of the stress tensor $\sigma_{\text{diss}}^{ik}$ depend linearly on the components of the deformation rate tensor \dot{u}_{ik} [20].
- (iii) The conditions of quasistatic motion are provided, i.e. $\dot{\xi} \ll c$, $\tau_{\text{vis}} \ll \tau_c$ [16, 17] (here c is the speed of sound in the material of particles and τ_{vis} is the relaxation time of viscous processes in its bulk).

The constant A in Eq. (5) reads [16, 17]

$$A = \frac{1}{3} \frac{(3\eta_2 - \eta_1)^2}{(3\eta_2 + 2\eta_1)} \left[\frac{(1 - \nu^2)(1 - 2\nu)}{Y\nu^2} \right]. \tag{6}$$

where Y and ν are respectively the Young modulus and the Poisson ratio of the particle material and the viscous constants η_1, η_2 relate (linearly) the dissipative stress tensor $\sigma_{\text{diss}}^{ik}$ to the deformation rate tensor \dot{u}_{ik} [16, 17, 20].

From Eq. (5) follows that

$$\beta = 1 \quad \gamma = \alpha - 1. \quad (7)$$

Consider now a collision of three-dimensional spherical particles of radii R_1 and R_2 . The Hertz contact contact law gives for the elastic force [21]

$$F_{\text{el}} = \rho \xi^{3/2}, \quad \rho \equiv \frac{2Y}{3(1-\nu^2)} \sqrt{R^{\text{eff}}}, \quad (8)$$

where $R^{\text{eff}} \equiv R_1 R_2 / (R_1 + R_2)$. With the set of exponents, $\alpha = 3/2$, $\beta = 1$ and $\gamma = 1/2$, which generally follows from the basic laws of the viscoelastic collision, the condition for the constant restitution coefficient, Eq. (4), is obviously not satisfied. For spherical particles the restitution coefficient could be constant only for $\gamma = 1/4$; this, however, is not consistent with the collision laws. Instead one obtains the functional dependence

$$\epsilon = \epsilon \left(v_{\text{imp}}^{1/5} \right). \quad (9)$$

Note that this conclusion comes from the general analysis of viscoelastic collisions with no other assumptions needed. Therefore, the dependence of the restitution coefficient on the impact velocity, Eq. (9), is a natural property, provided the assumption on viscoelasticity holds true which is the case in a wide range of impact velocities (see discussion on page 102). We want to mention that the functional dependence Eq. (9) was already given in [22] using heuristic arguments.

Rigorous calculations [23] yield for the dependence of the restitution coefficient on the impact velocity:

$$\epsilon = 1 - C_1 A \kappa^{2/5} v_{\text{imp}}^{1/5} + C_2 A^2 \kappa^{4/5} v_{\text{imp}}^{2/5} \mp \dots \quad (10)$$

with

$$\kappa = \left(\frac{3}{2} \right)^{3/2} \frac{Y \sqrt{R^{\text{eff}}}}{m^{\text{eff}} (1 - \nu^2)} \quad (11)$$

where $m^{\text{eff}} = m_1 m_2 / (m_1 + m_2)$ ($m_{1/2}$ are the masses of the colliding particles). Numerical values for the constants C_1 and C_2 obtained in [23] may be also written in a more convenient form [13]:

$$C_1 = \frac{\Gamma(3/5) \sqrt{\pi}}{2^{1/5} 5^{2/5} \Gamma(21/10)} = 1.15344, \quad C_2 = \frac{3}{5} C_1^2. \quad (12)$$

Although the next-order coefficients of the above expansion $C_3 = -0.483582$, $C_4 = 0.285279$, are now available [13], we assume that the dissipative constant A is small enough to ignore these high-order terms. (For large A a very accurate Padé approximation for $\epsilon(v_{\text{imp}})$ has been proposed recently [13]).

3 Time-Evolution of Temperature and of the Velocity Distribution Function

We consider a granular gas composed of N identical particles confined in a volume Ω . The particles are assumed to be smooth spheres, so that the collision properties are determined by the normal component of the relative motion only. The gas is supposed to be dilute enough so that one can assume binary collisions (i.e. neglect multiple collisions) and ignore the collision duration as compared with the mean free time in between successive collisions. We assume that the initial velocities of the particles (more precisely the temperature, which we define below) are not very large to assure viscoelastic properties of the collisions, i.e. to avoid plastic deformations and fragmentation. The final velocities are assumed not to be very small which allows to neglect surface forces as adhesion and others. Under these restrictions one can apply the viscoelastic collision model. Furthermore, we assume that dissipation is not large, so that the second-order expansion (10) for $\epsilon(v_{\text{imp}})$ describes the collisions accurately. We analyze the granular gas in the regime of homogeneous cooling, i.e. in the pre-clustering regime, when the gas is homogeneously distributed in space.

The impact velocity v_{imp} of colliding smooth spheres, which determines the value of the restitution coefficient according to Eq. (10), is given by the normal component of the relative velocity

$$v_{\text{imp}} = |\mathbf{v}_{12} \cdot \mathbf{e}| \quad \text{with} \quad \mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2. \quad (13)$$

The unit vector $\mathbf{e} = \mathbf{r}_{12}/|\mathbf{r}_{12}|$ gives the direction of the intercenter vector $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ at the instant of the collision.

The evolution of the granular gas proceeds by elementary collision events in which the pre-collisional velocities of colliding particles $\mathbf{v}_1, \mathbf{v}_2$, are converted into after-collisional ones, $\mathbf{v}_1^*, \mathbf{v}_2^*$, according to the rules

$$\begin{aligned} \mathbf{v}_1^* &= \mathbf{v}_1 - \frac{1}{2} [1 + \epsilon(|\mathbf{v}_{12} \cdot \mathbf{e}|)] (\mathbf{v}_{12} \cdot \mathbf{e}) \mathbf{e} \\ \mathbf{v}_2^* &= \mathbf{v}_2 + \frac{1}{2} [1 + \epsilon(|\mathbf{v}_{12} \cdot \mathbf{e}|)] (\mathbf{v}_{12} \cdot \mathbf{e}) \mathbf{e}, \end{aligned} \quad (14)$$

where ϵ depends on the impact velocity $v_{\text{imp}} = |\mathbf{v}_{12} \cdot \mathbf{e}|$. Due to the *direct* collision (14) the population in the velocity phase-space near the points $\mathbf{v}_1, \mathbf{v}_2$ decreases, while near the points $\mathbf{v}_1^*, \mathbf{v}_2^*$ it increases. The decrease of the population near $\mathbf{v}_1, \mathbf{v}_2$ caused by the direct collision, is (partly) counter-balanced by its increase in the *inverse* collision, where the after-collisional velocities are $\mathbf{v}_1, \mathbf{v}_2$ with the pre-collisional ones $\mathbf{v}_1^{**}, \mathbf{v}_2^{**}$. The rules for the inverse collision read

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_1^{**} - \frac{1}{2} [1 + \epsilon(|\mathbf{v}_{12}^{**} \cdot \mathbf{e}|)] (\mathbf{v}_{12}^{**} \cdot \mathbf{e}) \mathbf{e} \\ \mathbf{v}_2 &= \mathbf{v}_2^{**} + \frac{1}{2} [1 + \epsilon(|\mathbf{v}_{12}^{**} \cdot \mathbf{e}|)] (\mathbf{v}_{12}^{**} \cdot \mathbf{e}) \mathbf{e}. \end{aligned} \quad (15)$$

Note that in contrast to the case of $\epsilon = \text{const}$, the restitution coefficients in the inverse and in the direct collisions are different.

The Enskog-Boltzmann equation describes the evolution of the population of particles in the phase space on the mean-field level. The evolution is characterized by the distribution function $f(\mathbf{r}, \mathbf{v}, t)$, which for the force-free case does not depend on \mathbf{r} and obeys the equation [4, 24]

$$\begin{aligned} \frac{\partial}{\partial t} f(\mathbf{v}_1, t) &= g_2(\sigma) \sigma^2 \int d\mathbf{v}_2 \int d\mathbf{e} \Theta(-\mathbf{v}_{12} \cdot \mathbf{e}) |\mathbf{v}_{12} \cdot \mathbf{e}| \\ &\times \{ \chi f(\mathbf{v}_1^{**}, t) f(\mathbf{v}_2^{**}, t) - f(\mathbf{v}_1, t) f(\mathbf{v}_2, t) \} \equiv g_2(\sigma) I(f, f) \end{aligned} \quad (16)$$

where σ is the diameter of the particles. The contact value of the pair distribution function [25]

$$g_2(\sigma) = (2 - \eta)/2(1 - \eta)^3, \quad (17)$$

accounts on the mean-field level for the increasing frequency of collisions due to excluded volume effects with $\eta = \frac{1}{6} \pi n \sigma^3$ being the volume fraction.

The first term in the curled brackets in the right-hand side of Eq. (16) refers to the ‘‘gain’’ term for the population in the phase-space near the point \mathbf{v}_1 , while the second one is the ‘‘loss’’ term. The Heaviside function $\Theta(-\mathbf{v}_{12} \cdot \mathbf{e})$ discriminates approaching particles (which do collide) from separating particles (which do not collide), and $|\mathbf{v}_{12} \cdot \mathbf{e}|$ gives the length of the collision cylinder. Integration in Eq. (16) is performed over all velocities \mathbf{v}_2 and interparticle vectors \mathbf{e} in the direct collision. Equation (16) accounts also for the inverse collisions via the factor χ , which appears due to the Jacobian of the transformation $\mathbf{v}_1^{**}, \mathbf{v}_2^{**} \rightarrow \mathbf{v}_1, \mathbf{v}_2$, and due to the difference between the lengths of the collision cylinders of the direct and the inverse collision:

$$\chi = \frac{\mathcal{D}(\mathbf{v}_1^{**}, \mathbf{v}_2^{**})}{\mathcal{D}(\mathbf{v}_1, \mathbf{v}_2)} \frac{|\mathbf{v}_{12}^{**} \cdot \mathbf{e}|}{|\mathbf{v}_{12} \cdot \mathbf{e}|}. \quad (18)$$

For constant restitution coefficient the factor χ is a constant

$$\chi = \frac{1}{\epsilon^2} = \text{const}, \quad (19)$$

while for $\epsilon = \epsilon(v_{\text{imp}})$, as given in Eq. (10), it reads [26]

$$\chi = 1 + \frac{11}{5} C_1 A \kappa^{2/5} |\mathbf{v}_{12} \cdot \mathbf{e}|^{1/5} + \frac{66}{25} C_1^2 A^2 \kappa^{4/5} |\mathbf{v}_{12} \cdot \mathbf{e}|^{2/5} + \dots \quad (20)$$

From Eq. (20) it follows that $\chi = \chi(|\mathbf{v}_{12} \cdot \mathbf{e}|)$. Since the average velocity in granular gases changes with time, such a dependence of χ means, as we will show below, that χ and, therefore, the velocity distribution function itself depend explicitly on time. The time dependence of χ changes drastically the properties of the collision integral and destroys the simple scaling form of

the velocity distribution function, which holds for the case of the constant restitution coefficient (e.g. [4, 5]).

Nevertheless, some important properties of the collision integral are preserved. Namely, it may be shown that the relation

$$\begin{aligned} \frac{d}{dt} \langle \psi(t) \rangle &= \int d\mathbf{v}_1 \psi(\mathbf{v}_1) \frac{\partial}{\partial t} f(\mathbf{v}_1, t) = g_2(\sigma) \int d\mathbf{v}_1 \psi(\mathbf{v}_1) I(f, f) = \\ &= \frac{g_2(\sigma)\sigma^2}{2} \int d\mathbf{v}_1 d\mathbf{v}_2 \int d\mathbf{e} \Theta(-\mathbf{v}_{12} \cdot \mathbf{e}) |\mathbf{v}_{12} \cdot \mathbf{e}| f(\mathbf{v}_1, t) f(\mathbf{v}_2, t) \Delta[\psi(\mathbf{v}_1) + \psi(\mathbf{v}_2)] \end{aligned} \quad (21)$$

holds true, where

$$\langle \psi(t) \rangle \equiv \int d\mathbf{v} \psi(\mathbf{v}) f(\mathbf{v}, t) \quad (22)$$

is the average of some function $\psi(\mathbf{v})$, and

$$\Delta\psi(\mathbf{v}_i) \equiv [\psi(\mathbf{v}_i^*) - \psi(\mathbf{v}_i)] \quad (23)$$

denotes the change of $\psi(\mathbf{v}_i)$ in a direct collision.

Now we introduce the temperature of the three-dimensional granular gas,

$$\frac{3}{2} n T(t) = \int d\mathbf{v} \frac{m v^2}{2} f(\mathbf{v}, t), \quad (24)$$

where n is the number density of granular particles ($n = N/\Omega$), and the characteristic velocity $v_0^2(t)$ is related to temperature via

$$T(t) = \frac{1}{2} m v_0^2(t). \quad (25)$$

First we try the scaling ansatz

$$f(\mathbf{v}, t) = \frac{n}{v_0^3(t)} \tilde{f}(\mathbf{c}) \quad (26)$$

where $\mathbf{c} \equiv \mathbf{v}/v_0(t)$ and following [4, 7] assume that deviations from the Maxwellian distribution are not large, so that $\tilde{f}(\mathbf{c})$ may be expanded into a convergent series with the leading term being the Maxwellian distribution $\phi(c) \equiv \pi^{-3/2} \exp(-c^2)$. It is convenient to use the Sonine polynomials expansion [4, 7]

$$\tilde{f}(\mathbf{c}) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p S_p(c^2) \right\}. \quad (27)$$

These polynomials are orthogonal, i.e.

$$\int d\mathbf{c} \phi(c) S_p(c^2) S_{p'}(c^2) = \delta_{pp'} \mathcal{N}_p, \quad (28)$$

where $\delta_{pp'}$ is the Kronecker delta and \mathcal{N}_p is the normalization constant. The first few polynomials read

$$\begin{aligned} S_0(x) &= 1 \\ S_1(x) &= -x^2 + \frac{3}{2} \\ S_2(x) &= \frac{x^2}{2} - \frac{5x}{2} + \frac{15}{8}. \end{aligned} \quad (29)$$

Writing the Enskog-Boltzmann equation in terms of the scaling variable \mathbf{c}_1 , one observes that the factor χ may not be expressed only in terms of the scaling variable, but it depends also on the characteristic velocity $v_0(t)$, and thus depends on time. Therefore, the collision integral also occurs to be time-dependent. As a result, it is not possible to reduce the Enskog-Boltzmann equation to a pair of equations, one for the time evolution of the temperature and another for the time-independent scaling function, whereas for $\epsilon = \text{const.}$ the Boltzmann-Enskog equation is separable, e.g. [4, 5, 7]. Formally adopting the approach of Refs. [4, 7] for $\epsilon = \text{const.}$, one would obtain time-dependent coefficients a_p of the Sonine polynomials expansion. This means that the simple scaling ansatz (26) is violated for the case of the impact-velocity dependent restitution coefficient.

Thus, it seems natural to write the distribution function in the following general form

$$f(\mathbf{v}, t) = \frac{n}{v_0^3(t)} \tilde{f}(\mathbf{c}, t) \quad (30)$$

with

$$\tilde{f}(\mathbf{c}) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p(t) S_p(c^2) \right\} \quad (31)$$

and find then equations for the *time-dependent* coefficients $a_p(t)$. Substituting (30) into the Boltzmann equation (16) we obtain

$$\frac{\mu_2}{3} \left(3 + c_1 \frac{\partial}{\partial c_1} \right) \tilde{f}(\mathbf{c}, t) + B^{-1} \frac{\partial}{\partial t} \tilde{f}(\mathbf{c}, t) = \tilde{I}(\tilde{f}, \tilde{f}) \quad (32)$$

with

$$B = B(t) \equiv v_0(t) g_2(\sigma) \sigma^2 n. \quad (33)$$

We define the dimensionless collision integral:

$$\begin{aligned} \tilde{I}(\tilde{f}, \tilde{f}) &= \\ &\int d\mathbf{c}_2 \int d\mathbf{e} \Theta(-\mathbf{c}_{12} \cdot \mathbf{e}) |\mathbf{c}_{12} \cdot \mathbf{e}| \left\{ \tilde{\chi} \tilde{f}(\mathbf{c}_1^{**}, t) \tilde{f}(\mathbf{c}_2^{**}, t) - \tilde{f}(\mathbf{c}_1, t) \tilde{f}(\mathbf{c}_2, t) \right\} \end{aligned} \quad (34)$$

with the reduced factor $\tilde{\chi}$

$$\tilde{\chi} = 1 + \frac{11}{5} C_1 \delta' |\mathbf{c}_{12} \cdot \mathbf{e}|^{1/5} + \frac{66}{25} C_1^2 \delta'^2 |\mathbf{c}_{12} \cdot \mathbf{e}|^{2/5} + \dots \quad (35)$$

which depends now on time via a quantity

$$\delta'(t) \equiv A \kappa^{2/5} [2T(t)]^{1/10} \equiv \delta [2T(t)/T_0]^{1/10} . \quad (36)$$

Here $\delta \equiv A \kappa^{2/5} [T_0]^{1/10}$, T_0 is the initial temperature, and for simplicity we assume the unit mass, $m = 1$. We also define the moments of the dimensionless collision integral

$$\mu_p \equiv - \int d\mathbf{c}_1 c_1^p \tilde{I}(\tilde{f}, \tilde{f}) , \quad (37)$$

so that the second moment describes the rate of the temperature change:

$$\frac{dT}{dt} = -\frac{2}{3} B T \mu_2 . \quad (38)$$

Equation (38) follows from the definitions of the temperature and of the moment μ_2 . Note that these moments depend on time, in contrast to the case of the constant restitution coefficient, where these moments are time-independent [4].

Multiplying both sides of Eq. (32) with c_1^p and integrating over $d\mathbf{c}_1$, we obtain

$$\frac{\mu_2}{3} p \langle c^p \rangle - B^{-1} \sum_{k=1}^{\infty} \dot{a}_k \nu_{kp} = \mu_p \quad (39)$$

where integration by parts has been performed and we define

$$\nu_{kp} \equiv \int \phi(c) c^p S_k(c^2) d\mathbf{c} \quad (40)$$

$$\langle c^p \rangle \equiv \int c^p \tilde{f}(\mathbf{c}, t) d\mathbf{c} . \quad (41)$$

The calculation of ν_{kp} is straightforward; the first few of these read: $\nu_{22} = 0$, $\nu_{24} = \frac{15}{4}$. The odd moments $\langle c^{2n+1} \rangle$ vanish, while the even ones $\langle c^{2n} \rangle$ may be expressed in terms of a_k with $0 \leq k \leq n$, namely, $\langle c^2 \rangle = \frac{3}{2} - \frac{3}{2} a_1$. On the other hand, from the definition of temperature and of the thermal velocity in Eqs. (25) and (24) follows that $\langle c^2 \rangle = \frac{3}{2}$ and thus, $a_1 = 0$. Similar considerations yield $\langle c^4 \rangle = \frac{15}{4} (1 + a_2)$. The moments μ_p may be expressed in terms of coefficients a_2, a_3, \dots too; therefore, the system Eq. (39) is an infinite (but closed) set of equations for these coefficients.

It is not possible to get a general solution of the problem. However, since the dissipative parameter δ is supposed to be small, the deviations from the

Maxwellian distribution are presumably small too. Thus, we assume that one can neglect all high-order terms with $p > 2$ in the expansion (31). Then Eq. (39) is an equation for the coefficient a_2 . For $p = 2$ Eq. (39) converts into an identity, since $\langle c^2 \rangle = \frac{3}{2}$, $a_1 = 0$, $\nu_{22} = 0$ and $\nu_{24} = \frac{15}{4}$. For $p = 4$ we obtain

$$\dot{a}_2 - \frac{4}{3}B\mu_2(1 + a_2) + \frac{4}{15}B\mu_4 = 0. \quad (42)$$

In Eq. (42) B depends on time as

$$B(t) = (8\pi)^{-1/2}\tau_c(0)^{-1}[T(t)/T_0]^{1/2}, \quad (43)$$

where $\tau_c(0)$ is related to the initial mean-collision time,

$$\tau_c(0)^{-1} = 4\pi^{1/2}g_2(\sigma)\sigma^2nT_0^{1/2}. \quad (44)$$

The time evolution of the temperature is determined by Eq. (38), i.e. by the time dependence of μ_2 .

The time-dependent coefficients $\mu_p(t)$ may be expressed in terms of a_2 owing to their definition Eq. (37) and the approximation $\tilde{f} = \phi(c)[1 + a_2(t)S_2(c^2)]$. We finally obtain:

$$\begin{aligned} \mu_p = & -\frac{1}{2} \int d\mathbf{c}_1 \int d\mathbf{c}_2 \int d\mathbf{e} \Theta(-\mathbf{c}_{12} \cdot \mathbf{e}) |\mathbf{c}_{12} \cdot \mathbf{e}| \phi(c_1) \phi(c_2) \\ & \times \{1 + a_2 [S_2(c_1^2) + S_2(c_2^2)] + a_2^2 S_2(c_1^2) S_2(c_2^2)\} \Delta(c_1^p + c_2^p) \end{aligned} \quad (45)$$

with the definition of $\Delta(c_1^p + c_2^p)$ given above. Calculations performed up to the second order in terms of the dissipative parameter δ yield [26]:

$$\mu_2 = \sum_{k=0}^2 \sum_{n=0}^2 \mathcal{A}_{kn} \delta'^k a_2^n \quad (46)$$

where

$$\begin{aligned} \mathcal{A}_{00} &= 0; & \mathcal{A}_{01} &= 0; & \mathcal{A}_{02} &= 0 \\ \mathcal{A}_{10} &= \omega_0; & \mathcal{A}_{11} &= \frac{6}{25}\omega_0; & \mathcal{A}_{12} &= \frac{21}{2500}\omega_0 \\ \mathcal{A}_{20} &= -\omega_1; & \mathcal{A}_{21} &= -\frac{119}{400}\omega_1; & \mathcal{A}_{22} &= -\frac{4641}{640000}\omega_1 \end{aligned} \quad (47)$$

with

$$\omega_0 \equiv 2\sqrt{2\pi}2^{1/10}\Gamma\left(\frac{21}{10}\right)C_1 = 6.48562\dots \quad (48)$$

$$\omega_1 \equiv \sqrt{2\pi}2^{1/5}\Gamma\left(\frac{16}{5}\right)C_1^2 = 9.28569\dots \quad (49)$$

Similarly

$$\mu_4 = \sum_{k=0}^2 \sum_{n=0}^2 \mathcal{B}_{kn} \delta'^k a_2^n \tag{50}$$

with

$$\begin{aligned} \mathcal{B}_{00} &= 0; & \mathcal{B}_{01} &= 4\sqrt{2\pi}; & \mathcal{B}_{02} &= \frac{1}{8}\sqrt{2\pi} \\ \mathcal{B}_{10} &= \frac{56}{10}\omega_0; & \mathcal{B}_{11} &= \frac{1806}{250}\omega_0; & \mathcal{B}_{12} &= \frac{567}{12500}\omega_0 \\ \mathcal{B}_{20} &= -\frac{77}{10}\omega_1; & \mathcal{B}_{21} &= -\frac{149054}{13750}\omega_1; & \mathcal{B}_{22} &= -\frac{348424}{5500000}\omega_1 \end{aligned} \tag{51}$$

Thus, Eqs. (38) and (42), together with Eqs. (46) and (50) form a closed set to find the time evolution of the temperature and the coefficient a_2 . We want to stress an important difference for the time evolution of temperature for the case of the impact-velocity dependent restitution coefficient, as compared to that of a constant restitution coefficient. In the former case it is coupled to the time evolution of the coefficient a_2 , while in the latter case there is no such coupling since $a_2 = \text{const}$. This coupling may lead in to a rather peculiar time-dependence of the temperature.

Introducing the reduced temperature $u(t) \equiv T(t)/T_0$ we recast the set of equations (38) and (42) into the following form:

$$\begin{aligned} \dot{u} + \tau_0^{-1} u^{8/5} \left(\frac{5}{3} + \frac{2}{5} a_2 + \frac{7}{500} a_2^2 \right) \\ - \tau_0^{-1} q_1 \delta u^{17/10} \left(\frac{5}{3} + \frac{119}{240} a_2 + \frac{1547}{128000} a_2^2 \right) = 0 \end{aligned} \tag{52}$$

$$\dot{a}_2 - r_0 u^{1/2} \mu_2 (1 + a_2) + \frac{1}{5} r_0 u^{1/2} \mu_4 = 0, \tag{53}$$

where we introduce the characteristic time

$$\tau_0^{-1} = \frac{16}{5} q_0 \delta \cdot \tau_c(0)^{-1} \tag{54}$$

with

$$q_0 = 2^{1/5} \Gamma(21/10) C_1 / 8 = 5^{-2/5} \sqrt{\pi} \Gamma(3/5) / 8 = 0.173318 \dots \tag{55}$$

$$r_0 \equiv \frac{2}{3\sqrt{2\pi}} \tau_c(0)^{-1} \tag{56}$$

$$q_1 \equiv 2^{1/10} (\omega_1 / \omega_0) = 1.53445 \dots \tag{57}$$

As shown below the characteristic time τ_0 describes the time evolution of the temperature. To obtain these equations we use the expressions for $\mu_2(t)$, $B(t)$, and for the coefficients \mathcal{A}_{nk} . Note that the characteristic time τ_0 is $\delta^{-1} \gg 1$ times larger than the collision time $\sim \tau_c(0)$.

We will find the solution to these equations as expansions in terms of the small dissipative parameter δ ($\delta'(t) = \delta \cdot 2^{1/10} u^{1/10}(t)$):

$$u = u_0 + \delta \cdot u_1 + \delta^2 \cdot u_2 + \dots \quad (58)$$

$$a_2 = a_{20} + \delta \cdot a_{21} + \delta^2 \cdot a_{22} + \dots \quad (59)$$

Substituting Eqs. (46,50,58,59) into Eqs. (52,53), one can solve these equations perturbatively, for each order of δ . The solution of the order of $\mathcal{O}(1)$ reads for the coefficient $a_2(t)$ [26]:

$$a_{20}(t) \approx a_{20}(0) e^{-4t/(5\tau_E(0))}, \quad (60)$$

where $\tau_E = \frac{3}{2}\tau_c$ is the Enskog relaxation time, so that $a_{20}(t)$ vanishes for $t \sim \tau_0$. This refers to the relaxation of an initially non-Maxwellian velocity distribution to the Maxwellian distribution. Note that the relaxation occurs within few collisions per particle, similarly to the relaxation of common molecular gases.

We now assume that the initial distribution is Maxwellian, i.e., that $a_{20}(0) = 0$ for $t = 0$. Then the deviation from the Maxwellian distribution originates from the inelasticity of the interparticle collisions. For the case of $a_{20}(0) = 0$ the solution of the order of $\mathcal{O}(1)$ for the reduced temperature reads

$$\frac{T(t)}{T_0} = u_0(t) = \left(1 + \frac{t}{\tau_0}\right)^{-5/3}, \quad (61)$$

which coincides with the time-dependence of the temperature obtained previously using scaling arguments [23] (up to a constant τ_0 which may not be determined by scaling arguments).

The solution for $a_2(t)$ in linear approximation with respect to δ reads

$$a_2(t) = \delta \cdot a_{21}(t) = -\frac{12}{5} w(t)^{-1} \{ \text{Li}[w(t)] - \text{Li}[w(0)] \} \quad (62)$$

where

$$w(t) \equiv \exp \left[(q_0 \delta)^{-1} (1 + t/\tau_0)^{1/6} \right]. \quad (63)$$

and with the logarithmic Integral

$$\text{Li}(x) = \int_0^x \frac{1}{\ln(t)} dt. \quad (64)$$

For $t \ll \tau_0$ the coefficient $a_2(t)$ (62) reduces to

$$a_2(t) = -\delta \cdot h \left(1 - e^{-4t/(5\tau_E(0))} \right) \quad (65)$$

where

$$h \equiv 2^{1/10} (\mathcal{B}_{10} - 5\mathcal{A}_{10}) / 16\pi = (3/10)\Gamma(21/10)2^{1/5}C_1 = 0.415964. \quad (66)$$

As it follows from Eq. (65), after a transient time of the order of few collisions per particle, i.e. for $\tau_E(0) < t \ll \tau_0$, $a_2(t)$ saturates at the “steady-state”-value $-h\delta = -0.415964\delta$, i.e. it changes only slowly on the time-scale $\sim \tau_c(0)$. On the other hand, for $t \gg \tau_0$ one obtains

$$a_2(t) \simeq -\delta \cdot h (t/\tau_0)^{-1/6} \quad (67)$$

so that $a_2(t)$ decays to zero on a time-scale $\sim \tau_0$, i.e. slowly in the collisional time-scale $\sim \tau_c(0) \ll \tau_0$. The velocity distribution thus tends asymptotically to the Maxwellian distribution. For this regime the first-order correction for the reduced temperature, $u_1(t)$, reads [26]:

$$u_1(t) = \left(\frac{12}{25}h + 2q_1 \right) (t/\tau_0)^{-11/6} = 3.26856 (t/\tau_0)^{-11/6}, \quad (68)$$

where we used the above results for the constants h and q_1 . From the last equation one can see how the coupling between the temperature evolution and the evolution of the velocity distribution influences the evolution of temperature. Indeed, if there were no such coupling, there would be no coupling term in Eq. (52), and thus no contribution from $\frac{12}{25}h$ to the prefactor of $u_1(t)$ in Eq. (68). This would noticeably change the time behavior of $u_1(t)$. On the other hand, the leading term in the time dependence of temperature, $u_0(t)$, is not affected by this kind of coupling.

In Fig. 2 and Fig. 3 we show the time dependence of the coefficient $a_2(t)$ of the Sonine polynomial expansion and of the temperature of the granular gas. The analytical findings are compared with the numerical solution of the system (52,53). As one can see from the figures the analytical theory reproduces fairly well the numerical solution for the case of small δ .

As it follows from Fig. 2 (where the time is given in collisional units), for small δ the following scenario of evolution of the velocity distribution takes place for a force-free granular gas. The initial Maxwellian distribution evolves to a non-Maxwellian distribution, with the discrepancy between these two characterized by the second coefficient of the Sonine polynomials expansion a_2 . The deviation from the Maxwellian distribution (described by a_2) quickly grows, until it saturates after a few collisions per particle at a “steady-state” value. At this instant the deviation from the Maxwellian distribution is maximal, with the value $a_2 \approx -0.4\delta$ (Fig. 2a). This refers to the first “fast” stage of the evolution, which takes place on a mean-collision

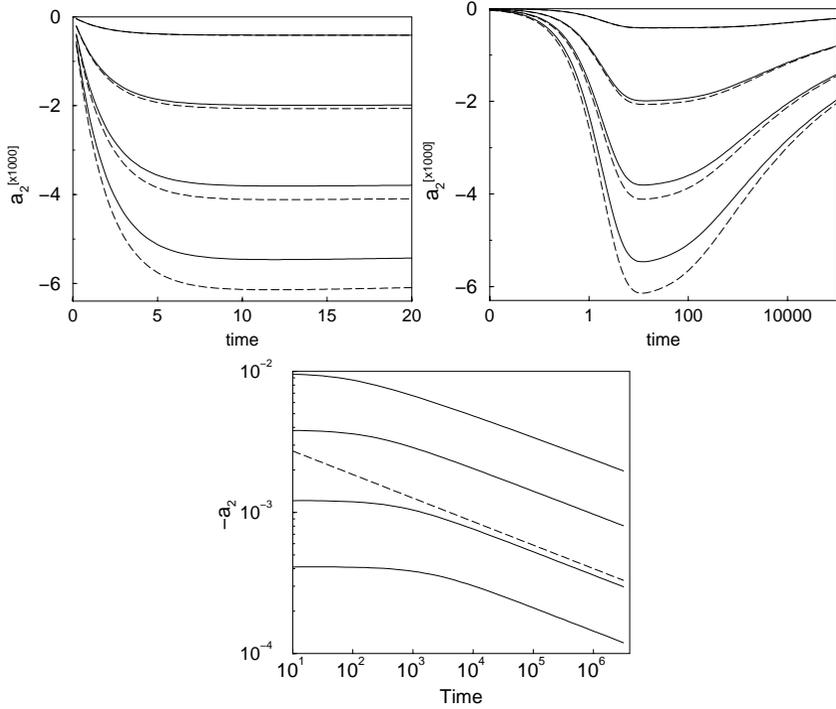


Fig. 2. Time dependence of the second coefficient of the Sonine polynomial expansion $a_2(t)$. Time is given in units of the mean collisional time $\tau_c(0)$. (Left): $a_2 \times 1000$ (solid lines) for $\delta = 0.001, 0.005, 0.01, 0.015$ (top to bottom) together with the linear approximation (dashed lines); (Right): the same as (left) but for larger times; (Middle): $-a_2(t)$ over time (log-scale) for $\delta = 0.03, 0.01, 0.003, 0.001$ (top to bottom) together with the power-law asymptotics $\sim t^{-1/6}$.

time-scale $\sim \tau_c(0)$. After this maximal deviation is reached, the second “slow” stage of the evolution starts. At this stage a_2 decays to zero on the “slow” time scale $\tau_0 \sim \delta^{-1}\tau_c(0) \gg \tau_0(0)$, which corresponds to the time scale of the temperature evolution (Fig. 2b); the decay of the coefficient $a_2(t)$ in this regime occurs according to a power law $\sim t^{-1/6}$ (Fig. 2c). Asymptotically the Maxwellian distribution would be achieved, if the clustering process did not occur.

Fig. 3 illustrates the significance of the first-order correction $u_1(t)$ in the time-evolution of temperature. This becomes more important as the dissipation parameter δ grows (Figs. 3a,b). At large times the results of the first-order theory (with $u_1(t)$ included) practically coincide with the numerical results, while zero-order theory (without $u_1(t)$) demonstrates noticeable deviations (Fig. 3c).

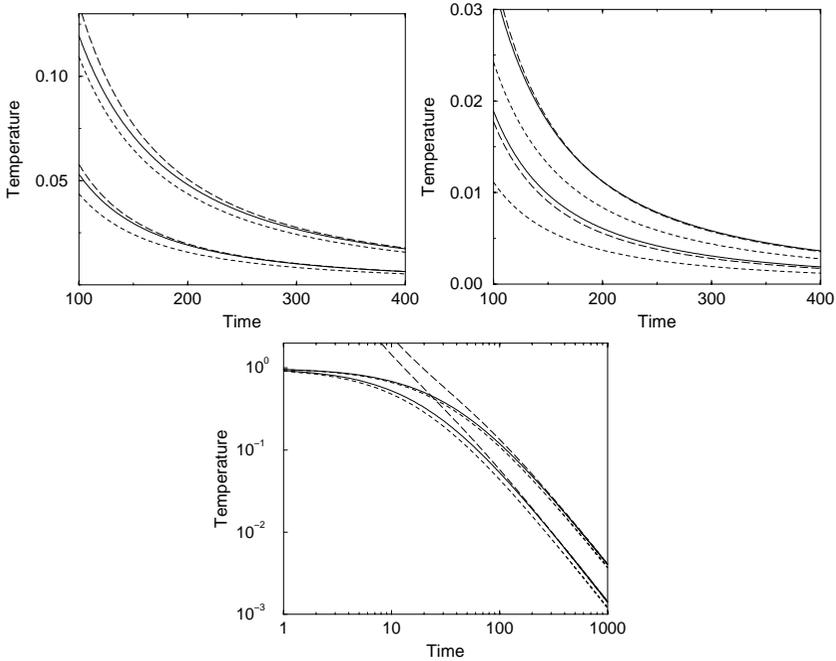


Fig. 3. Time-evolution of the reduced temperature, $u(t) = T(t)/T_0$. The time is given in units of mean collisional time $\tau_c(0)$. Solid line: numerical solution, short-dashed: $u_0(t) = (1 + t/\tau_0)^{-5/3}$ (zero-order theory), long-dashed: $u(t) = u_0(t) + \delta u_1(t)$ (first-order theory). (Left): for $\delta = 0.05, 0.1$ (top to bottom); (Right): $\delta = 0.15, 0.25$ (top to bottom); (Middle): the same as (Left) but log-scale and larger ranges.

For larger values of δ the linear theory breaks down. Unfortunately, the equations obtained for the second order approximation $\mathcal{O}(\delta^2)$ are too complicated to be treated analytically. Hence, we studied them only numerically (see Fig. 4). As compared to the case of small δ , an additional intermediate regime in the time-evolution of the velocity distribution is observed. The first “fast” stage of evolution takes place, as before, on the time scale of few collisions per particle, where maximal deviation from the Maxwellian distribution is achieved (Fig. 4). For $\delta \geq 0.15$ these maximal values of a_2 are positive. Then, on the second stage (intermediate regime), which continues 10 – 100 collisions, a_2 changes its sign and reaches a maximal negative deviation. Finally, on the third, slow stage, $a_2(t)$ relaxes to zero on the slow time-scale $\sim \tau_0$, just as for small δ . In Fig. 4 we show the first stage of the time evolution of $a_2(t)$ for systems with large δ . At a certain value of the dissipative parameter δ the behavior changes qualitatively, i.e. the system then reveals another time scale as discussed above.

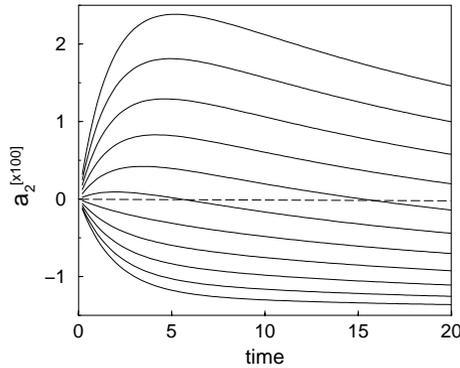


Fig. 4. Time dependence of the second coefficient of the Sonine polynomial expansion $a_2(t) \times 100$. Time is given in units of mean collisional time $\tau_c(0)$. $\delta = 0.1, 0.11, 0.12, \dots, 0.20$ (bottom to top).

Figure 5 shows the numerical solution of Eqs. (52) and (53) for the second Sonine coefficient $a_2(t)$ as a function of time. One can clearly distinguish the different stages of evolution of the velocity distribution function. A more detailed investigation of the evolution of the distribution function for larger dissipation is subject of present research [26].

The interesting property of the granular gases in the regime of homogeneous cooling is the overpopulation of the high-velocity tails in the velocity distribution [5], which has been shown for granular gases consisting of particles which interact via a constant restitution coefficient, $\epsilon = \text{const}$. How does the velocity dependence of the restitution coefficient as it appears for viscoelastic spheres influence this effect? We observe, that for the case of $\epsilon = \epsilon(v_{\text{imp}})$ the functional form (i.e. the exponential overpopulation [5]) persists, but it decreases with time on the “slow” time-scale $\sim \tau_0$. Namely we obtain for the velocity distribution for $c \gg 1$ [26]:

$$\tilde{f}(\mathbf{c}, t) \sim \exp \left[-\frac{b}{\delta} c \left(1 + \frac{t}{\tau_0} \right)^{1/6} \right]. \quad (69)$$

where $b = \sqrt{\pi/2} (16q_0/5)^{-1} = 2.25978 \dots$, which holds for $t \gg \tau_c(0)$. Again we see that the distribution tends asymptotically to the Maxwellian distribution, since the overpopulation vanishes as $t \rightarrow \infty$.

Using the temperature and the velocity distribution of a granular gas as were derived in this section, one can calculate the kinetic coefficients. In the next section we consider the simplest one – the self-diffusion coefficient.

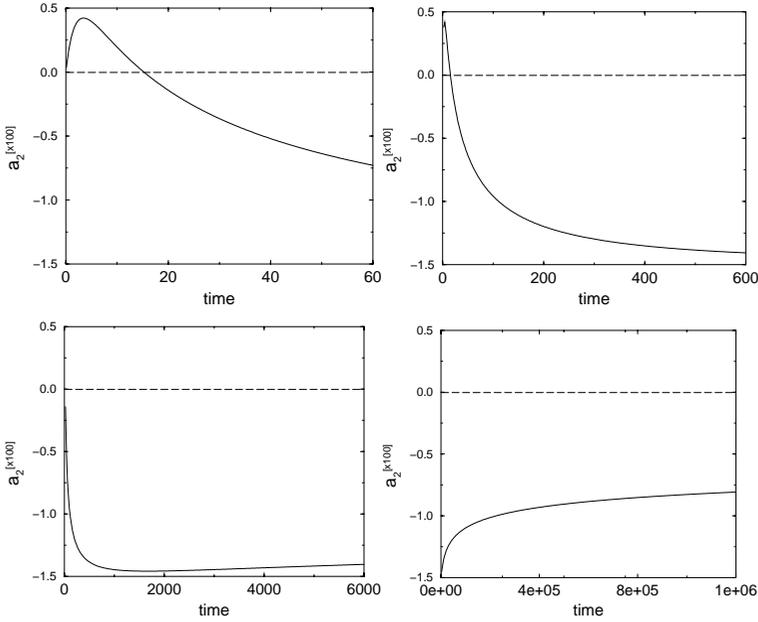


Fig. 5. The second Sonine coefficient a_2 for $\delta = 0.16$ over time. The numerical solutions of Eqs. (52) and (53) show all stages of evolution discussed in the text.

4 Self-Diffusion in Granular Gases of Viscoelastic Particles

In the simplest case diffusion of particles occurs when there are density gradients in the system. The diffusion coefficient D relates the flux of particles \mathbf{J} to the density gradient ∇n according to a linear relation, provided the gradients are not too large:

$$\mathbf{J} = -D\nabla n. \tag{70}$$

The coefficient D also describes the statistical average of the migration of a single particle. For *equilibrium* 3D-systems the mean-square displacement of a particle reads

$$\langle (\Delta r(t))^2 \rangle_{\text{eq}} = 6 D t, \tag{71}$$

where $\langle \dots \rangle_{\text{eq}}$ denotes the *equilibrium* ensemble averaging. For *nonequilibrium* systems, such as granular gases, one should consider the time-dependent diffusion coefficient $D(t)$ and the corresponding generalization of Eq. (71):

$$\langle (\Delta r(t))^2 \rangle = 6 \int^t D(t') dt', \tag{72}$$

where $\langle \dots \rangle$ denotes averaging over the nonequilibrium ensemble. If the migration of a particle occurs in a uniform system composed of particles of the same kind, this process is called “self-diffusion”. Correspondingly, the kinetic coefficient D is called self-diffusion coefficient.

To find the mean-square displacement, one writes

$$\langle (\Delta r(t))^2 \rangle = \left\langle \int_0^t \mathbf{v}(t') dt' \int_0^t \mathbf{v}(t'') dt'' \right\rangle \quad (73)$$

and encounters then with the velocity autocorrelation function

$$K_v(t', t) \equiv \langle \mathbf{v}(t') \mathbf{v}(t'') \rangle$$

which should be evaluated in order to obtain the mean-square displacement and the self-diffusion coefficient.

To calculate $K_v(t', t)$ we use the approximation of uncorrelated successive binary collision, which is valid for moderately dense systems, and an approach based on the formalism of the pseudo-Liouville operator \mathcal{L} [27]. The pseudo-Liouville operator is defined as

$$i\mathcal{L} = \sum_j \mathbf{v}_j \cdot \frac{\partial}{\partial \mathbf{r}_j} + \sum_{i < j} \hat{T}_{ij}. \quad (74)$$

The first sum in (74) refers to the free streaming of the particles (the ideal part) while the second sum refers to the particle interactions which are described by the binary collision operators [28]

$$\hat{T}_{ij} = \sigma^2 \int d\mathbf{e} \Theta(-\mathbf{v}_{ij} \cdot \mathbf{e}) |\mathbf{v}_{ij} \cdot \mathbf{e}| \delta(\mathbf{r}_{ij} - \sigma \mathbf{e}) (\hat{b}_{ij}^{\mathbf{e}} - 1), \quad (75)$$

where $\Theta(x)$ is the Heaviside function. The operator $\hat{b}_{ij}^{\mathbf{e}}$ is defined as

$$\hat{b}_{ij}^{\mathbf{e}} f(\mathbf{r}_i, \mathbf{r}_j, \mathbf{v}_i, \mathbf{v}_j \dots) = f(\mathbf{r}_i, \mathbf{r}_j, \mathbf{v}_i^*, \mathbf{v}_j^* \dots), \quad (76)$$

where f is some function of the dynamical variables and \mathbf{v}_i^* and \mathbf{v}_j^* are the postcollisional velocities from Eq. (14). The pseudo-Liouville operator gives the time derivative of any dynamical variable B (e.g. [24]):

$$\frac{d}{dt} B(\{\mathbf{r}_i, \mathbf{v}_i\}, t) = i\mathcal{L} B(\{\mathbf{r}_i, \mathbf{v}_i\}, t). \quad (77)$$

Therefore, the time evolution of B reads ($t > t'$)

$$B(\{\mathbf{r}_i, \mathbf{v}_i\}, t) = e^{i\mathcal{L}(t-t')} B(\{\mathbf{r}_i, \mathbf{v}_i\}, t'). \quad (78)$$

With Eq. (78) the time-correlation function reads

$$\langle B(t') B(t) \rangle = \int d\Gamma \rho(t') B(t') e^{i\mathcal{L}(t-t')} B(t), \quad (79)$$

where $\int d\Gamma$ denotes integration over all degrees of freedom and $\rho(t')$ depends on temperature T , density n , etc., which change on a time-scale $t \gg \tau_c$.

Now we assume that

- (i) the coordinate part and the velocity part of the distribution function $\rho(t)$ factorize, and
- (ii) the molecular chaos hypothesis is valid.

This suggests the following form of the distribution function:

$$\rho(t) = \rho(\mathbf{r}_1, \dots, \mathbf{r}_N) \cdot f(\mathbf{v}_1, t) \dots f(\mathbf{v}_N, t). \quad (80)$$

In accordance with the molecular chaos assumption the sequence of the successive collisions occurs without correlations. If the variable B does not depend on the positions of the particles, its time-correlation function decays exponentially [29]:

$$\langle B(t')B(t) \rangle = \langle B^2 \rangle_{t'} e^{-|t-t'|/\tau_B(t')} \quad (t > t'). \quad (81)$$

where $\langle \dots \rangle_{t'}$ denotes the averaging with the distribution function taken at time t' . The relaxation time τ_B is inverse to the initial slope of the autocorrelation function [29], as it may be found from the time derivative of $\langle B(t')B(t) \rangle$ taken at $t = t'$. Equations (79) and (81) then yield

$$-\tau_B^{-1}(t') = \int d\Gamma \rho(t') Bi\mathcal{L}B / \langle B^2 \rangle_{t'} = \frac{\langle Bi\mathcal{L}B \rangle_{t'}}{\langle B^2 \rangle_{t'}}. \quad (82)$$

The relaxation time $\tau_B^{-1}(t')$ depends on time via the distribution function $\rho(t')$ and varies on the time-scale $t \gg \tau_c$.

Let $B(t)$ be the velocity of some particle, say $\mathbf{v}_1(t)$. Then with $3T(t) = \langle v^2 \rangle_t$, Eqs. (81) and (82) (with Eqs. (74) and (75)) read [10]

$$\langle \mathbf{v}_1(t') \cdot \mathbf{v}_1(t) \rangle = 3T(t') e^{-|t-t'|/\tau_v(t')} \quad (83)$$

$$-\tau_v^{-1}(t') = (N-1) \frac{\langle \mathbf{v}_1 \cdot \hat{T}_{12} \mathbf{v}_1 \rangle_{t'}}{\langle \mathbf{v}_1 \cdot \mathbf{v}_1 \rangle_{t'}}. \quad (84)$$

To obtain Eq. (84) we take into account that $\mathcal{L}_0 \mathbf{v}_1 = 0$, $\hat{T}_{ij} \cdot \mathbf{v}_1 = 0$ (for $i \neq 1$) and the identity of the particles.

Straightforward calculation yields for the case of a constant restitution coefficient:

$$\tau_v^{-1}(t) = \frac{\epsilon + 1}{2} \frac{8}{3} n \sigma^2 g_2(\sigma) \sqrt{\pi T(t)} = \frac{\epsilon + 1}{2} \tau_E^{-1}(t), \quad (85)$$

where $\tau_E(t) = \frac{3}{2} \tau_c(t)$ is the Enskog relaxation time [24]. Note that according to Eq. (85), $\tau_v = \frac{2}{1+\epsilon} \tau_E > \tau_E$, i.e., the velocity correlation time for inelastic collisions exceeds that of elastic collisions. This follows from partial suppression of the backscattering of particles due to inelastic losses in their normal relative motion, which, thus, leads to more stretched particle trajectories, as compared to the elastic case.

Similar (although somewhat more complicated) computations may be performed for the system of viscoelastic particles yielding

$$\tau_v^{-1}(t) = \tau_E^{-1}(t) \left[1 + \frac{3}{16} a_2(t) - 4q_0 \delta u^{1/10}(t) \right], \quad (86)$$

where $q_0 = 0.173318$ has been already introduced and $a_2(t)$, $u(t)$ are the same as defined above. To obtain Eq. (86) we neglect terms of the order of $\mathcal{O}(a_2^2)$, $\mathcal{O}(\delta^2)$ and $\mathcal{O}(a_2 \delta)$.

Using the velocity correlation function one writes

$$\langle (\Delta r(t))^2 \rangle = 2 \int_0^t dt' 3T(t') \int_{t'}^t dt'' e^{-|t''-t'|/\tau_v(t')}. \quad (87)$$

On the short-time scale $t \sim \tau_c$, $T(t')$ and $\tau_v(t')$ may be considered as constants. Integrating in (87) over t'' and equating with (72) yields for $t \gg \tau_c \sim \tau_v$ the time-dependent self-diffusion coefficient

$$D(t) = T(t) \tau_v(t). \quad (88)$$

Substituting the dependencies for $u(t) = T(t)/T_0$ and $a_2(t)$ as functions of time, which has been derived in the previous section, we obtain the time dependence of the coefficient of self-diffusion $D(t)$. For $t \gg \tau_0$ this may be given in an explicit form:

$$\frac{D(t)}{D_0} \simeq \left(\frac{t}{\tau_0} \right)^{-5/6} + \delta \left(4q_0 + q_1 + \frac{21}{400} h \right) \left(\frac{t}{\tau_0} \right)^{-1}, \quad (89)$$

where the constants q_0 , q_1 and h are given above. Hence, the prefactor in the term proportional to δ reads $(4q_0 + q_1 + \frac{21}{400} h) = 2.24956$, and D_0 is the initial Enskog value of the self-diffusion coefficient

$$D_0^{-1} = \frac{8}{3} \pi^{1/2} n g_2(\sigma) \sigma^2 T_0^{-1/2}. \quad (90)$$

Correspondingly, the mean-square displacement reads asymptotically for $t \gg \tau_0$:

$$\langle (\Delta r(t))^2 \rangle \sim t^{1/6} + b \delta \log t + \dots, \quad (91)$$

where b is some constant. This dependence holds true for times

$$\tau_c(0) \delta^{-1} \ll t \ll \tau_c(0) \delta^{-11/5}. \quad (92)$$

The first inequality in Eq. (92) follows from the condition $\tau_0 \ll t$, while the second one follows from the condition $\tau_c(t) \ll \tau_0$, which means that temperature changes are slow on the collisional time-scale. For the constant restitution coefficient one obtains

$$T(t)/T_0 = [1 + \gamma_0 t / \tau_c(0)]^{-2}, \quad (93)$$

where $\gamma_0 \equiv (1 - \epsilon^2) / 6$ [3, 9]. Thus, using Eqs. (85) and (88) one obtains for the mean-square displacement in this case

$$\langle (\Delta r(t))^2 \rangle \sim \log t. \quad (94)$$

As it follows from Eqs. (91) and (94) the impact-velocity dependent restitution coefficient, Eq. (10), leads to a significant change of the long-time behavior of the mean-square displacement of particles in the laboratory-time. Compared to its logarithmically weak dependence for the constant restitution coefficient, the impact-velocity dependence of the restitution coefficient gives rise to a considerably faster increase of this quantity with time, according to a power law.

One can also compare the dynamics of the system in its inherent-time scale. First we consider the average cumulative number of collisions per particle $\mathcal{N}(t)$ as an inherent measure for time (e.g. [2, 8]). It may be found by integrating $d\mathcal{N} = \tau_c(t)^{-1} dt$ [9]. For a constant restitution coefficient ϵ one obtains $\mathcal{N}(t) \sim \log t$, while for the impact-velocity dependent $\epsilon(v_{\text{imp}})$ one has $\mathcal{N}(t) \sim t^{1/6}$. Therefore, the temperature and the mean-square displacement behave in these cases as

$\epsilon = \text{const}$	$\epsilon = \epsilon(v_{\text{imp}})$
$T(\mathcal{N}) \sim e^{-2(1-\epsilon^2)\mathcal{N}}$	$T(\mathcal{N}) \sim \mathcal{N}^{-10}$
$\langle (\Delta r(\mathcal{N}))^2 \rangle \sim \mathcal{N}$	$\langle (\Delta r(\mathcal{N}))^2 \rangle \sim \mathcal{N}$

If the number of collisions per particle $\mathcal{N}(t)$ would be the relevant quantity specifying the stage of the granular gas evolution, one would conjecture that the dynamical behavior of a granular gas with a constant ϵ and velocity-dependent ϵ are identical, provided an \mathcal{N} -based time-scale is used. Whereas in equilibrium systems the number of collisions is certainly an appropriate measure of time, in nonequilibrium systems this value has to be treated with more care. As a trivial example may serve a particle bouncing back and forth between two walls, each time it hits a wall it loses part of its energy: If one describes this system using a \mathcal{N} -based time, one would come to the conclusion that the system conserves its energy, which is certainly not the proper description of the system. According to our understanding, therefore,

the number of collision is not an appropriate time scale to describe physical reality. Sometimes, it may be even misleading.

Indeed, as it was shown in Ref. [8], the value of \mathcal{N}_c , corresponding to a crossover from the linear regime of evolution (which refers to the homogeneous cooling state) to the nonlinear regime (when clustering starts) may differ by orders of magnitude, depending on the restitution coefficient and on the density of the granular gas. Therefore, to analyze the behavior of a granular gas, one can try an alternative inherent time-scale, $\mathcal{T}^{-1} \equiv T(t)/T_0$ which is based on the gas temperature. Given two systems of granular particles at the same density and the same initial temperature T_0 , consisting of particles colliding with constant and velocity-dependent restitution coefficient, respectively, the time \mathcal{T} allows to compare directly their evolution. A strong argument to use a temperature-based time has been given by Goldhirsch and Zanetti [3] who have shown that as the temperature decays, the evolution of the system changes from a linear regime to a nonlinear one. Recent numerical results of Ref. [8] also support our assumption: It was shown that while \mathcal{N}_c differs by more than a factor of three for two different systems, the values of \mathcal{T}_c , (defined, as $\mathcal{T}_c = T(\mathcal{N}_c)/T_0$) are very close [8]. These arguments show that one could consider \mathcal{T} as a relevant time-scale to analyze the granular gas evolution.

With the temperature decay $T(\mathcal{N})/T_0 \sim e^{-2\gamma_0\mathcal{N}}$ for a constant restitution coefficient and $T(\mathcal{N})/T_0 \sim \mathcal{N}^{-10}$ for the impact-velocity dependent one, we obtain the following dependencies:

$\epsilon = \text{const}$	$\epsilon = \epsilon(v_{\text{imp}})$
$T \sim \frac{1}{\mathcal{T}}$	$T \sim \frac{1}{\mathcal{T}}$
$\langle (\Delta r(\mathcal{T}))^2 \rangle \sim \log \mathcal{T}$	$\langle (\Delta r(\mathcal{T}))^2 \rangle \sim \mathcal{T}^{1/10}$

This shows that in the temperature-based time-scale, in which the cooling of both systems is synchronized, the mean-square displacement grows logarithmically slow for the case of constant restitution coefficient and much faster, as a power law, for the system of viscoelastic particles with $\epsilon = \epsilon(v_{\text{imp}})$. Thus, we conclude that clustering may be retarded for the latter system.

5 Conclusion

In conclusion, we considered kinetic properties of granular gases composed of viscoelastic particles, which implies the impact-velocity dependence of the restitution coefficient. We found that such dependence gives rise to some new effects in granular gas dynamics: (i) complicated, non-monotonous time-dependence of the coefficient a_2 of the Sonine polynomial expansion, which

describes the deviation of the velocity distribution from the Maxwellian and (ii) enhanced spreading of particles, which depends on time as a power law, compared to a logarithmically weak dependence for the systems with a constant ϵ .

The Table below compares the properties of granular gases consisting of particles interacting via a constant coefficient of restitution $\epsilon = \text{const}$ and consisting of viscoelastic particles where the collisions are described using an impact velocity dependent restitution coefficient $\epsilon = \epsilon(v_{\text{imp}})$:

$\epsilon = \text{const}$	$\epsilon = \epsilon(v_{\text{imp}})$
ϵ is a model parameter	$\epsilon = 1 - C_1 A \kappa^{2/5} v_{\text{imp}}^{1/5} + \dots$ $C_1 = 1.15396, C_2 = \frac{3}{5} C_1^2, \dots$ $\kappa = \kappa(Y, \nu, m, R)$ $A = A(\eta_1, \eta_2, Y, \nu)$ all quantities are defined via parameters of the particle material $Y, \nu, \eta_{1/2}$ and their mass and radius.
Small parameter	
$1 - \epsilon^2$ – does not depend on the state of the system	$\delta = A \kappa^{2/5} T_0^{1/10}$ – depends on the initial temperature T_0 .
Temperature	
$T = T_0 (1 + t/\tau_0)^{-2}$	$T = T_0 (1 + t/\tau_0)^{-5/3}$
Velocity distribution	
$f(\mathbf{v}, t) = \frac{n}{v_0^3(t)} \tilde{f}(\mathbf{c})$ $\tilde{f}(\mathbf{c}, t) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p S_p(c^2) \right\}$ $a_2 = \text{const.}$	$f(\mathbf{v}, t) = \frac{n}{v_0^3(t)} \tilde{f}(\mathbf{c}, t)$ $\tilde{f}(\mathbf{c}) = \phi(c) \left\{ 1 + \sum_{p=1}^{\infty} a_p(t) S_p(c^2) \right\}$ $a_2 = a_2(t)$ – is a (complicated) function of time.
Self-diffusion	
$\langle (\Delta r(t))^2 \rangle \sim \log t$	$\langle (\Delta r(t))^2 \rangle \sim t^{1/6}$

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