1 Collision of Adhesive Viscoelastic Particles

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Abstract

The collision of convex bodies is considered for small impact velocity, when plastic deformation and fragmentation may be disregarded. In this regime the contact is governed by forces according to viscoelastic deformation and by adhesion. The viscoelastic interaction is described by a modified Hertz law, while for the adhesive interactions, the model by Johnson, Kendall and Roberts (JKR) is adopted. We solve the general contact problem of convex viscoelastic bodies in quasi-static approximation, which implies that the impact velocity is much smaller than the speed of sound in the material and that the viscosity relaxation time is much smaller than the duration of a collision. We estimate the threshold impact velocity which discriminates restitutive and sticking collisions. If the impact velocity is not large as compared with the threshold velocity, adhesive interaction becomes important, thus limiting the validity of the pure viscoelastic collision model.

1.1 Introduction

The large set of phenomena observed in granular systems, ranging from sand and powders on Earth to granular gases in planetary rings and protoplanetary discs, is caused by the specific particle interaction. Besides elastic forces, common for molecular or atomic materials (solids, liquids and gases), colliding granular particles also exert dissipative forces. These forces correspond to the dissipation of mechanical energy in the bulk of the grain material as well as on their surfaces. The dissipated energy transforms into energy of the internal degrees of freedom of the grains, that is, the particles are heated. In many applications, however, the increase in temperature of the particle material may be neglected (see, e.g. [6]).

The dynamical properties of granular materials depend sensitively on the details of the dissipative forces acting between contacting grains. Therefore, choosing the appropriate model of the dissipative interaction is crucial for an adequate description of these systems. In real granular systems the particles may have a complicated non-spherical shape, they may be non-uniform and even composed of smaller grains, kept together by adhesion. The particles may differ in size, mass and in their material properties. In what follows we consider the contact of granular particles under simplifying conditions. We assume that the particles are smooth, convex and of uniform material. The latter assumption allows us to describe the particle deformation by continuum mechanics, disregarding their molecular structure.

It is assumed that particles exert forces on each other exclusively via pairwise mechanical contact, i.e., electromagnetic interaction and gravitational attraction are not considered.

The Physics of Granular Media.
Edited by Haye Hinrichsen, Dietrich E. Wolf

ISBN: 3-527-40373-6

1.2 Forces Between Granular Particles

1.2.1 Elastic Forces

When particles deform each other due to a static (or quasi-static) contact they experience an elastic interaction force. Elastic deformation implies that, after separation of the contacting particles, they recover their initial shape, i.e., there is no plastic deformation. The stress tensor $\sigma_{\text{el}}^{ij}(\vec{r}\,)$ describes the i-component of the force, acting on a unit surface which is normal to the direction j $(i, j = \{x, y, z\})$. In the elastic regime the stress is related to the material deformation

$$u_{ij}(\vec{r}\,) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) , \tag{1.1}$$

where $\vec{u}(\vec{r}\,)$ is the displacement field at the point $\vec{r}$ in the deformed body, via the linear relation

$$\sigma_{\text{el}}^{ij}(\vec{r}\,) = E_1\left(u_{ij}(\vec{r}\,) - \frac{1}{3}\delta_{ij}u_{ll}(\vec{r}\,)\right) + E_2\delta_{ij}u_{ll}(\vec{r}\,) . \tag{1.2}$$

Repeated indices are implicitly summed over (Einstein convention). The coefficients E_1 and E_2 read

$$E_1 = \frac{Y}{(1+\nu)}, \qquad E_2 = \frac{Y}{3(1-2\nu)} , \tag{1.3}$$

where Y is the Young modulus and ν is the Poisson ratio. Let the pressure $\vec{P}(x, y)$ act on the surface of an elastic semispace, $z > 0$, leading to a displacement field in the bulk of the semispace [18]:

$$u_i = \iint G_{ik}\left(x - x', y - y', z\right) P_k\left(x', y'\right) dx'dy' , \tag{1.4}$$

where $G_{ik}\left(x, y, z\right)$ is the corresponding Green function. For the contact problem addressed here we need only the z-component of the displacement on the surface $z = 0$, that is, we need only the component

$$G_{zz}(x, y, z = 0) = \frac{(1-\nu^2)}{\pi Y}\frac{1}{\sqrt{x^2+y^2}} = \frac{(1-\nu^2)}{\pi Y}\frac{1}{r} \tag{1.5}$$

of the Green function [18].

Consider a contact of two convex smooth bodies labeled as 1 and 2. We assume that only normal forces, with respect to the contact area, act between the particles. In the contact region their surfaces are flat. For the coordinate system centered in the middle of the contact region, where $x = y = z = 0$, the following relation holds true:

$$B_1x^2 + B_2y^2 + u_{z1}(x, y) + u_{z2}(x, y) = \xi , \tag{1.6}$$

where u_{z1} and u_{z2} are respectively the z-components of the displacement in the material of the first and of the second bodies on the plane $z = 0$. The sum of the compressions of both bodies in the center of the contact area defines ξ. The constants B_1 and B_2 are related to the radii of curvature of the surfaces in contact [18]:

$$\begin{aligned} 2\,(B_1 + B_2) &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R'_1} + \frac{1}{R'_2} \\ 4\,(B_1 - B_2)^2 &= \left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 + \left(\frac{1}{R'_1} - \frac{1}{R'_2}\right)^2 + 2\cos 2\varphi \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{R'_1} - \frac{1}{R'_2}\right). \end{aligned} \tag{1.7}$$

Here R_1, R_2 and R'_1, R'_2 are respectively the principal radii of curvature of the first and the second body at the point of contact and φ is the angle between the planes corresponding to the curvature radii R_1 and R'_1. Equations (1.6), (1.7) describe the general case of the contact between two smooth bodies (see [18] for details). The physical meaning of (1.6) is easy to see for the case of a contact of a soft sphere of a radius R ($R_1 = R_2 = R$) with a hard, undeformed plane ($R'_1 = R'_2 = \infty$). In this case $B_1 = B_2 = 1/R$, the compressions of the sphere and of the plane are respectively $u_{z1}(0,0) = \xi$ and $u_{z2} = 0$, and the surface of the sphere before the deformation is given by $z(x,y) = (x^2 + y^2)/R$. Then (1.6) reads in the flattened area $u_{z1}(x,y) = \xi - z(x,y)$, that is, it gives the condition for a point $z(x,y)$ on the body's surface to touch the plane $z = 0$.

The displacements u_{z1} and u_{z2} may be expressed in terms of the normal pressure $P_z(x,y)$ which acts between the compressed bodies in the plane $z = 0$. Using (1.4) and (1.5) we rewrite (1.6) as

$$\frac{1}{\pi}\left(\frac{1-\nu_1^2}{Y_1} + \frac{1-\nu_2^2}{Y_2}\right)\iint \frac{P_z(x',y')}{r}dx'dy' = \xi - B_1 x^2 - B_2 y^2\,, \tag{1.8}$$

where $r = \sqrt{(x-x')^2 + (y-y')^2}$ and integration is performed over the contact area. Equation (1.8) is an integral equation for the unknown function $P_z(x,y)$. We compare this equation with the mathematical identity [18]

$$\iint \frac{dx'dy'}{r}\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}} = \frac{\pi ab}{2}\int\limits_0^\infty \left[1 - \frac{x^2}{a^2+t} - \frac{y^2}{b^2+t}\right], \frac{dt}{\sqrt{(a^2+t)(b^2+t)t}} \tag{1.9}$$

where integration is performed over the elliptical area $x'^2/a^2 + y'^2/b^2 = 1$. The left-hand sides of both equations contain integrals of the same type, while the right-hand sides contain quadratic forms of the same type. Therefore, the contact area is an ellipse with the semi-axes a and b and the pressure is of the form $P_z(x,y) = \mathrm{const}\sqrt{1 - x^2/a^2 - y^2/b^2}$. The constant may be found from the total elastic force F_{el} acting between the bodies. Integrating $P_z(x,y)$ over the contact area we obtain

$$P_z(x,y) = \frac{3F_{\mathrm{el}}}{2\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,. \tag{1.10}$$

We substitute (1.10) into (1.8) and replace the double integration over the contact area by integration over the variable t, according to the identity (1.9). Thus, we obtain an equation containing terms proportional to x^2, y^2 and a constant. Equating the corresponding coefficients we obtain

$$\xi = \frac{F_{\rm el} D}{\pi} \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)t}} = \frac{F_{\rm el} D}{\pi} \frac{N(x)}{b} \tag{1.11}$$

$$B_1 = \frac{F_{\rm el} D}{\pi} \int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)t}} = \frac{F_{\rm el} D}{\pi} \frac{M(x)}{a^2 b} \tag{1.12}$$

$$B_2 = \frac{F_{\rm el} D}{\pi} \int_0^\infty \frac{dt}{(b^2+t)\sqrt{(a^2+t)(b^2+t)t}} = \frac{F_{\rm el} D}{\pi} \frac{M(1/x)}{a b^2}\,, \tag{1.13}$$

where

$$D \equiv \frac{3}{4}\left(\frac{1-\nu_1^2}{Y_1} + \frac{1-\nu_2^2}{Y_2}\right) \tag{1.14}$$

and $x \equiv a^2/b^2$ is the ratio of the contact ellipse semi-axes. In (1.11)-(1.13) we introduce the short-hand notations[1]

$$N(x) = \int_0^\infty \frac{dt}{\sqrt{(1+xt)(1+t)t}} \tag{1.15}$$

$$M(x) = \int_0^\infty \frac{dt}{(1+t)\sqrt{(1+t)(1+xt)t}}\,. \tag{1.16}$$

From these relations will follow the size of the contact area, a, b, and the compression ξ as functions of the elastic force $F_{\rm el}$ and the geometrical coefficients B_1 and B_2.

The dependence of the force $F_{\rm el}$ on the compression ξ may be obtained from scaling arguments. If we rescale $a^2 \to \alpha a^2$, $b^2 \to \alpha b^2$, $\xi \to \alpha \xi$ and $F_{\rm el} \to \alpha^{3/2} F_{\rm el}$, with α constant, Eqs. (1.11)–(1.13) remain unchanged. That is, when ξ changes by the factor α, the semi-axis a and b change by the factor $\alpha^{1/2}$ and the force by the factor $\alpha^{3/2}$, i.e., $a \sim \xi^{1/2}$, $b \sim \xi^{1/2}$ and

$$F_{\rm el} = {\rm const}\, \xi^{3/2}\,. \tag{1.17}$$

The dependence (1.17) holds true for all smooth convex bodies in contact. To find the constant in (1.17) we divide (1.13) by (1.12) and obtain the transcendental equation

$$\frac{B_2}{B_1} = \frac{\sqrt{x} M\left(1/x\right)}{M(x)} \tag{1.18}$$

for the ratio of semi-axes x. Let x_0 be the root of Eq. (1.18), then $a^2 = x_0 b^2$ and we obtain

$$\xi = \frac{F_{\rm el} D}{\pi} \frac{N(x_0)}{b} \tag{1.19}$$

$$B_1 = \frac{F_{\rm el} D}{\pi} \frac{M(x_0)}{x_0 b^3}\,, \tag{1.20}$$

[1] The function $N(x)$ and $M(x)$ may be expressed as a combination of the Jacobian elliptic functions $E(x)$ and $K(x)$ [1].

where $N(x_0)$ and $M(x_0)$ are pure numbers. Equations (1.19), (1.20) allow us to find the semi-axes b and the elastic force $F_{\rm el}$ as functions of the compression ξ. Hence we obtain the force, i.e., we get the constant in (1.17):

$$F_{\rm el} = \frac{\pi}{D}\left(\frac{M(x_0)}{B_1 x_0 N(x_0)}\right)^{1/2} \xi^{3/2} = C_0 \xi^{3/2} \,. \tag{1.21}$$

For the special case of contacting spheres ($a = b$), the constants B_1 and B_2 read

$$B_1 = B_2 = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\frac{1}{R^{\rm eff}} \,. \tag{1.22}$$

In this case $x_0 = 1$, $N(1) = \pi$, and $M(1) = \pi/2$, leading to the solution of (1.19), (1.20):

$$a^2 = R^{\rm eff}\,\xi \tag{1.23}$$

$$F_{\rm el} = \rho \xi^{3/2}\,; \qquad \rho \equiv \frac{2Y}{3(1-\nu^2)}\sqrt{R^{\rm eff}} \,, \tag{1.24}$$

where we use the definition (1.14) of the constant D. This contact problem was solved by Heinrich Hertz in 1882 [14]. It describes the force between *elastic* particles. For inelastically deforming particles it describes the repulsive force in the static case.

1.2.2 Viscous Forces

When the contacting particles move with respect to each other, i.e., the deformation changes with time, an additional dissipative force arises, which acts in the opposite direction to the relative particle motion. The dissipative processes occurring in the bulk of the body cause a viscous contribution to the stress tensor. For small deformation the respective component of the stress tensor is proportional to the deformation rate $\dot{u}_{ij}(\vec{r})$, according to the general relation [8]:

$$\sigma^{ij}_{\rm dis}(\vec{r},t) = E_1 \int_0^t d\tau \psi_1(t-\tau)\left[\dot{u}_{ij}(\vec{r},\tau) - \frac{1}{3}\delta_{ij}\dot{u}_{ll}(\vec{r},\tau)\right] + E_2 \int_0^t d\tau \psi_2(t-\tau)\delta_{ij}\dot{u}_{ll}(\vec{r},\tau)\,, \tag{1.25}$$

where the (dimensionless) functions $\psi_1(t)$ and $\psi_2(t)$ are the relaxation functions for the distortion deformation and $\psi_2(t)$ for the dilatation deformation.

In many important applications the viscous stress tensor may be simplified significantly. If the relative velocity of the colliding bodies is much smaller than the speed of sound in the particle material and if the characteristic relaxation times of the dissipative processes $\tau_{\rm vis,\,1/2}$ are much smaller than the duration of the collision t_c,

$$\tau_{\rm vis,\,1/2} \equiv \int_0^\infty \psi_{1/2}(\tau)d\tau \ll t_c \,, \tag{1.26}$$

the viscous constants η_1 and η_2 may be used instead of the functions $\psi_1(t)$ and $\psi_2(t)$. Thus

$$\eta_{1/2} = E_{1/2}\tau_{\rm vis,\,1/2} = E_{1/2}\int_0^\infty \psi_{1/2}(\tau)d\tau \tag{1.27}$$

and the dissipative stress tensor reads (see [8] for details)

$$\sigma_{\rm dis}^{ij}(\vec{r},t) = \eta_1 \left[\dot{u}_{ij}(\vec{r},\tau) - \frac{1}{3}\delta_{ij}\dot{u}_{ll}(\vec{r},\tau) \right] + \eta_2 \delta_{ij}\dot{u}_{ll}(\vec{r},\tau)\,. \tag{1.28}$$

It may be also shown that the above conditions are equivalent to the assumption of quasi-static deformation [8, 7]. When the material is deformed quasi-statically, the displacement field $\vec{u}(\vec{r})$ in the particles coincides with that for the static case $\vec{u}_{\rm el}(\vec{r})$, which is the solution of the elastic contact problem. The field $\vec{u}_{\rm el}(\vec{r})$, in its turn, is completely determined by the compression ξ, which varies with time during the collision, i.e., $\vec{u}_{\rm el} = \vec{u}_{\rm el}(\vec{r},\xi)$. Therefore, the corresponding displacement rate may be approximated as

$$\dot{\vec{u}}(\vec{r},t) \simeq \dot{\xi}\frac{\partial}{\partial \xi}\vec{u}_{\rm el}(\vec{r},\xi) \tag{1.29}$$

and the dissipative stress tensor reads, respectively

$$\sigma_{\rm dis}^{ij} = \dot{\xi}\frac{\partial}{\partial \xi}\left[\eta_1 \left(u_{ij}^{\rm el} - \frac{1}{3}\delta_{ij}u_{ll}^{\rm el} \right) + \eta_2 \delta_{ij} u_{ll}^{\rm el} \right]\,. \tag{1.30}$$

From (1.30) and (1.2) follows the relation between the elastic and dissipative stress tensors within the quasi-static approximation,

$$\sigma_{\rm dis}^{ij} = \dot{\xi}\frac{\partial}{\partial \xi}\sigma_{\rm el}^{ij}\left(E_1 \leftrightarrow \eta_1, E_2 \leftrightarrow \eta_2\right)\,, \tag{1.31}$$

where we emphasize that the expression for the dissipative tensor may be obtained from the corresponding expression for the elastic tensor after substituting the elastic constants by the relative viscous constants, and application of the operator $\dot{\xi}\partial/\partial\xi$.

The component $\sigma_{\rm el}^{zz}$ of the elastic stress is equal to the normal pressure P_z at the plane $z = 0$ of the elastic problem, Eq. (1.10)

$$\begin{aligned} \sigma_{\rm el}^{zz}(x,y,0) =& E_1 \frac{\partial u_z}{\partial z} + \left(E_2 - \frac{E_1}{3} \right)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \\ =& \frac{3F_{\rm el}}{2\,\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,. \end{aligned} \tag{1.32}$$

Now we compute the total dissipative force acting between the bodies. Instead of a direct computation of the dissipative stress tensor, we employ the method proposed in [8, 7]: We transform the coordinate axes as

$$x = \alpha x'\,, \qquad y = \alpha y'\,, \qquad z = z' \tag{1.33}$$

with

$$\alpha = \left(\frac{\eta_2 - \frac{1}{3}\eta_1}{\eta_2 + \frac{2}{3}\eta_1} \right)\left(\frac{E_2 + \frac{2}{3}E_1}{E_2 - \frac{1}{3}E_1} \right) \qquad \beta = \frac{(\eta_2 - \frac{1}{3}\eta_1)}{\alpha(E_2 - \frac{1}{3}E_1)} \tag{1.34}$$

$$a = \alpha a' \qquad b = \alpha b'\,. \tag{1.35}$$

and perform the transformations

$$\begin{aligned}
&\eta_1 \frac{\partial u_z}{\partial z} + \left(\eta_2 - \frac{\eta_1}{3}\right)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
&= \beta \left[E_1 \frac{\partial u_z}{\partial z'} + \left(E_2 - \frac{E_1}{3}\right)\left(\frac{\partial u_x}{\partial x'} + \frac{\partial u_y}{\partial y'} + \frac{\partial u_z}{\partial z'}\right)\right] \\
&= \beta \frac{3F_{\rm el}}{2\,\pi a' b'} \sqrt{1 - \frac{x'^{\,2}}{a'^{\,2}} - \frac{y'^{\,2}}{b'^{\,2}}} = \beta\alpha^2 \frac{3F_{\rm el}}{2\,\pi ab} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,.
\end{aligned} \tag{1.36}$$

Applying the operator $\dot{\xi}\partial/\partial\xi$ to the last expression on the right-hand side we obtain the dissipative stress tensor. Subsequent integration over the contact area yields, finally, the total dissipative force acting between the bodies:

$$F_{\rm dis} = A\dot{\xi}\frac{\partial}{\partial\xi} F_{\rm el}(\xi)\,, \tag{1.37}$$

where

$$A \equiv \alpha^2\beta = \frac{1}{3}\frac{(3\eta_2 - \eta_1)^2}{(3\eta_2 + 2\eta_1)}\left[\frac{(1-\nu^2)(1-2\nu)}{Y\nu^2}\right]. \tag{1.38}$$

Using the scaling relations for the elastic force, Eq. (1.17), and for the semi-axes of the contact ellipse, we obtain

$$\frac{\partial F_{\rm el}}{\partial\xi} = \frac{3}{2}\frac{F_{\rm el}}{\xi}\,, \qquad \frac{\partial a}{\partial\xi} = \frac{1}{2}\frac{a}{\xi}\,, \qquad \frac{\partial b}{\partial\xi} = \frac{1}{2}\frac{b}{\xi}\,. \tag{1.39}$$

Then from (1.36) and (1.21), the distribution of the dissipative pressure in the contact area may be found:

$$P_z^{\rm dis}(x,y) = \frac{3A}{4\pi}\,\frac{AC_0}{ab}\dot{\xi}\sqrt{\xi}\,\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2}\,, \tag{1.40}$$

where the constant C_0 is defined in (1.21).

We wish to stress that, to derive the above expressions, we assumed only that the surfaces of the two bodies in the vicinity of the contact point before the deformation, are described by the quadratic forms $z_1 = \kappa_{ij}^{(1)} x_i x_j$ and $z_2 = \kappa^{(2)} x_i x_j$ $(i,j = x,y,z)$, where $\kappa_{ij}^{(1/2)}$ are symmetric tensors [18]. Therefore, the relations obtained are valid for a contact of arbitrarily shaped convex bodies. For spherical particles of identical material, (1.37) and (1.24) yield [8, 7]

$$F_{\rm dis} = \frac{3}{2}A\rho\dot{\xi}\sqrt{\xi}\,, \tag{1.41}$$

with ρ as defined in (1.24). Hence, the total force acting between viscoelastic spheres takes the simple form [8, 7]

$$F = \rho\left(\xi^{3/2} + \frac{3}{2}A\sqrt{\xi}\dot{\xi}\right)\,. \tag{1.42}$$

The range of validity of (1.42) for the viscoelastic force is determined by the quasi-static approximation. The impact velocity must be significantly smaller than the speed of sound. On the other hand, the impact velocity must not be too small in order to neglect adhesion. We also neglect plastic deformation in the material.

1.2.3 Adhesion of Contacting Particles

1.2.3.1 Models of Adhesive Interaction

The Hertz theory has been derived for the contact of non-adhesive particles. Adhesion becomes important when the distance of the particle surfaces approaches the range of molecular forces. Johnson, Kendall and Roberts (JKR) [17] extended the Hertz theory by taking into account adhesion in the flat contact region. They show that the contact area is enlarged by the action of the adhesive force. Therefore, they introduced an apparent Hertz load F_H which would cause this enlarged contact area. To simplify the notation, we consider the contact of identical spheres. The contact area is then a circle of radius a, which corresponds to the compression ξ_H for the Hertz load F_H. In reality, however, this contact radius occurs at the compression ξ which is smaller than ξ_H. In the JKR theory it is assumed that the difference between the Hertz compression ξ_H and the actual one, ξ, may be attributed to the additional stress

$$P_B(x,y) = \frac{F_B}{2\pi a^2}\left(1-\frac{r^2}{a^2}\right)^{-1/2}, \tag{1.43}$$

which is the solution of the classical Boussinesq problem [32]. This distribution of the normal surface traction gives rise to a constant displacement over a circular region of an elastic body. The displacement ξ_B corresponding to the contact radius a and the total load F_B are related by

$$\xi_B = \frac{2}{3} D \frac{F_B}{a}, \tag{1.44}$$

where the constant D is defined in (1.14).

The value of $F_B < 0$ mimics the additional surface forces, such that the pressure is positive (compressive) in the center of the contact area, while it is negative (tensile) near the boundary [17]. Hence, the shape of the body is determined by the action of two effective forces F_H and F_B. The total force between the particles is their difference, $F = F_H - F_B$. Johnson *et al.* assumed that the elastic energy stored in the deformed spheres may be found as a difference of the elastic energy corresponding to the Hertz force F_H and that due to the force F_B [17]. Using

$$U_s = -\pi\gamma a^2 \tag{1.45}$$

for the surface energy, where $\gamma > 0$ is twice the surface free energy per unit area of the solid in vacuum or gas, and minimizing the total energy, we find [17]

$$F_B = -2\pi a^2 \sqrt{\frac{3\gamma}{2\pi D a}}, \tag{1.46}$$

and, thus, the contact radius corresponding to the total force F:

$$a^3 = \frac{1}{2} D\, R \left(F + \frac{3}{2}\pi\gamma R + \sqrt{3\pi\gamma R F + \left(\frac{3}{2}\pi\gamma R\right)^2} \right) \tag{1.47}$$

and also the compression

$$\xi = \frac{2a^2}{R} - \sqrt{\frac{8\pi\gamma D a}{3}}\,. \tag{1.48}$$

The first term in (1.48) is the Hertz compression ξ_H, which coincides with (1.23) for $R^{\text{eff}} \to R/2$. Equation (1.47) may be solved to express the total force as a function of the contact radius:

$$F(a) = \frac{2a^3}{DR} - \sqrt{\frac{6\pi\gamma}{D}} a^{3/2}\,. \tag{1.49}$$

For vanishing applied load the contact radius a_0 is finite:

$$a_0^3 = \frac{3}{2} D\pi\gamma R^2\,. \tag{1.50}$$

For negative applied load the contact radius decreases and the condition for a real solution of (1.47) yields the maximal negative force which the adhesion forces can resist,

$$F_{\text{sep}} = -\frac{3}{4}\pi\gamma R\,, \tag{1.51}$$

corresponding to the contact radius

$$a_{\text{sep}}^3 = \frac{3}{8} D\pi\gamma R^2 = \frac{1}{4} a_0^3\,. \tag{1.52}$$

For a larger (in the absolute value) negative force, the spheres separate. For spheres of dissimilar radii, in (1.47)–(1.52) R should be substituted by $2R^{\text{eff}}$.

Another approach to the problem of the adhesive contact was developed by Derjaguin, Muller and Toporov (DMT). They assumed that the Hertz profile of the pressure distribution on the surface stays unaffected by adhesion and obtained the pull-off force $F_{\text{sep}} = -2\pi\gamma R^{\text{eff}}$ [9]. The assumption of the Hertz profile allows one to avoid the singularities of the pressure distribution (1.43) on the boundary of the contact zone. Since the experimental measurement of γ is problematic, it is not possible to check the validity of the JKR and DMT theories, i.e., to resolve their disagreement.

In later studies [20, 21] a more accurate theoretical analysis has been performed. The elastic equations have been solved numerically for a simplified microscopic model of adhesive surfaces with Lennard–Jones interaction. Within this microscopic approach, the relative accuracy of different theories has been estimated for a wide range of model parameters. It was found that the DMT theory is valid for small adhesion and for small, hard particles. JKR theory is more reliable for large, soft particles with large adhesion forces, which, however, should be short-ranged.

In [2] the Lennard–Jones continuum model of solids was studied. The adhesive forces between the surfaces then read

$$P_s(h) = \frac{H}{6\pi h^3}\left[\frac{z_0^6}{h^6} - 1\right] . \tag{1.53}$$

Here $P_s(h)$ describes the forces acting per unit area between the surfaces, $h = h(r)$ is the actual microscopic distance between them. H is the Hamaker constant, characterizing the van der Waals attraction of the particles in a gas or vacuum and z_0 is the equilibrium separations of the surfaces. The surface energy in this model is defined by

$$\gamma = \frac{H}{16\pi z_0^2} . \tag{1.54}$$

It was observed in [2] that the accuracies of different theories vary depending of the value of the Tabor parameter μ, [29]

$$\mu^{3/2} \equiv \frac{2}{3}\gamma D\sqrt{R^{\rm eff}/z_0^3} . \tag{1.55}$$

In agreement with [20, 21] it has been shown [2] that small values of μ (small hard particles with low surface energies) favor the DMT theory ($\mu < 10^{-2}$) while for $\mu \sim$ 1–10 the JKR theory proves to be rather more accurate. Both JKR and DMT fail for large μ when the strong adhesion is combined with the soft material of the contacting bodies. In this limit, the surfaces jump into contact, which corresponds to a spontaneous non-equilibrium transition (see e.g. [27]). Similar analysis has been performed later [11], where the author concluded that the DTM theory generally fails, both in original and corrected forms. One of the main conclusions of [2, 11] is that the JKR theory, albeit simple, gives relatively accurate predictions for basic quantities in the range of its validity ($\mu \sim$ 1–10).

Among the theories developed to cover the DMT–JKR transition regimes [20, 21, 11, 29, 16, 19] the theory by Maugis [19] is the most frequently used. It is based on a simplified model of adhesive forces. The adhesive force of constant intensity P_D is extended over a fixed distance h_D above the surface, yielding the surface tension $\gamma = P_D h_D$. The description of a contact in this model is based on two coupled analytical equations which are to be solved numerically. The recently developed double-Hertz model [12, 13] constructs the solution for the adhesive contact as a sum of two Hertzian solutions, which make the theory analytically more tractable than the Maugis model. Combining, in the adopted manner, the successful assumptions of the JKR and the modified DMT model, a generalized analytical theory for the adhesive contact has been proposed [26].

In what follows we assume that the parameters of our system belong to the range of validity of the JKR model, $\mu \sim 1 - 10$, and will use this simple analytical theory to describe the adhesive contacts between spheres. Moreover, we assume that the adhesive force is small. To estimate the influence of the adhesive force, we approximate $\xi \approx 2a^2/R$ in (1.48) and substitute it into (1.49) to obtain (see also [28]),

$$F \approx \rho\xi^{3/2} - \sqrt{6\pi\gamma/D}\left(R^{\rm eff}\right)^{3/4}\xi^{3/4} . \tag{1.56}$$

1.2.3.2 Viscoelasticity in Adhesive Interactions

The adhesive forces between particles cause the additional deformation in the contacting bodies as compared to a pure Hertzian deformation, hence in the corresponding dynamical problem an additional deformation rate arises. Therefore, the dissipative forces must have an additional component attributed to the adhesive interactions. The adhesive contact of viscoelastic spheres has been studied numerically in [28, 13]. In [5] the quasi-static condition for the colliding viscoelastic adhesive spheres was used and an analytical expression for the interaction force has been derived for the JKR model. Similar to the case for non-adhesive particles, it was assumed that in the quasi-static approximation, the deformation field may be parameterized by the value of the compression ξ. (Note that this assumption neglects the possible hysteresis which can happen for the negative total force [2]). Performing the same transformation which lead to the expression (1.37) for the case of non-adhesive contact, and using the approximation (1.56) we obtain the estimate for the dissipative forces [5]

$$F_{\rm dis} = \frac{3}{2}A\rho\dot{\xi}\sqrt{\xi} + \frac{3}{4}B\sqrt{6\pi\gamma/D}\left(R^{\rm eff}\right)^{3/4}\dot{\xi}\xi^{-1/4} \tag{1.57}$$

$$B \equiv \alpha\beta = \frac{(3\eta_2 - \eta_1)Y\nu}{3(1+\nu)(1-2\nu)}\,. \tag{1.58}$$

Note the singularity in the second term of (1.57) at $\xi = 0$ [2]. It is attributed to the quasi-static approximation for JKR theory and physically reflects the fact that the adhesive particles can jump into contact [27] with the discontinuous change of the compression ξ. Consider now how the above forces determine the particle dynamics.

1.3 Collision of Granular Particles

1.3.1 Coefficient of Restitution

Based on the particle interaction forces discussed so far, we turn now to the description of the particle collisions. It is assumed that the colliding particles do not exchange tangential forces[3], hence, only normal motion is considered. Let the particles be spheres of the same material, which start to collide at time $t = 0$ at relative normal velocity g (impact rate). The time-dependent compression then reads

$$\xi(t) = R_i + R_j - |\vec{r}_i(t) - \vec{r}_j(t)|\ , \tag{1.59}$$

where $\vec{r}_i(t)$ and $\vec{r}_j(t)$ are positions of the particle centers at time t (see Figure 1.1). The relative normal motion of particles at a collision is equivalent to the motion of a point particle with the effective mass

$$m^{\rm eff} = \frac{m_i m_j}{m_i + m_j}\,. \tag{1.60}$$

[2] This is a weak or integrable singularity, that is $\int_0^\epsilon \xi^{-1/4} d\xi \sim \epsilon^{3/4} \to 0$ for $\epsilon \to 0$. Hence for practical application of (1.57) one can use $\xi > \epsilon$, where ϵ may be very small but a finite number.

[3] See [6] for a discussion of the consistency of this assumption.

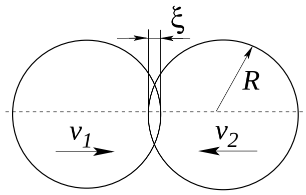

Figure 1.1: Head-on collision of identical spheres. The time-dependent state is characterized by the compression $\xi(t) \equiv 2R - |\vec{r}_1(t) - \vec{r}_2(t)|$ and the compression rate $\dot{\xi}(t) = v_1(t) - v_2(t)$.

For the moment let us neglect adhesion and consider the collision of viscoelastic particles interacting via the force (1.42). The equation of motion and the initial conditions read

$$\ddot{\xi} + \frac{\rho}{m^{\text{eff}}}\left(\xi^{3/2} + \frac{3}{2}A\sqrt{\xi}\dot{\xi}\right) = 0\,, \qquad \dot{\xi}(0) = g\,, \qquad \xi(0) = 0\,. \tag{1.61}$$

When granular particles collide, part of the energy of the relative motion is dissipated. The coefficient of (normal) restitution quantifies this phenomenon:

$$\varepsilon = -\dot{\xi}\left(t_c\right)/\dot{\xi}(0) = -\dot{\xi}(t_c)/g\,, \tag{1.62}$$

where $\dot{\xi}(0) = g$ is the pre-collision relative velocity and t_c is the duration of the collision. In general, ε is a function of the impact velocity. It can be obtained by integrating (1.61) numerically [15, 8, 7] or analytically [24].

1.3.2 Dimensional Analysis

The analytical solution [24] requires considerable efforts, here we give a simplified derivation which is based on a dimensional analysis of the equation of motion (1.61) [23]. This method was employed before [31] to prove that the frequent assumption ε =const. is inconsistent with mechanics of materials. For the dimensional analysis, the elastic and dissipative forces are represented in the more general form

$$F_{\text{el}} = m^{\text{eff}} D_1\, \xi^{\alpha}\,, \qquad F_{\text{diss}} = m^{\text{eff}} D_2\, \xi^{\gamma} \dot{\xi}^{\beta}\,, \tag{1.63}$$

with $D_{1/2}$ being material parameters. With these notations the equation of motion for colliding particles reads

$$\ddot{\xi} + D_1\, \xi^{\alpha} + D_2\, \xi^{\gamma}\, \dot{\xi}^{\beta} = 0\,, \qquad \dot{\xi}(0) = g\,, \qquad \xi(0) = 0\,. \tag{1.64}$$

For the case of pure elastic deformation ($D_2 = 0$) the maximal compression ξ_0 is obtained by equating the initial kinetic energy, $m^{\text{eff}} g^2/2$ and the elastic energy $m^{\text{eff}} D_1 \xi_0^{\alpha+1}/(\alpha+1)$:

$$\xi_0 \equiv \left(\frac{\alpha+1}{2\,D_1}\right)^{1/(1+\alpha)} g^{2/(1+\alpha)}\,. \tag{1.65}$$

We chose ξ_0 as the characteristic length of the problem. The time needed to cover the distance ξ_0 when traveling at velocity g defines the characteristic time: $\tau_0 \equiv \xi_0/g$. Thus, the dimensionless variables read

$$\hat{\xi} \equiv \xi/\xi_0\,, \qquad \dot{\hat{\xi}} \equiv \dot{\xi}/g\,, \qquad \ddot{\hat{\xi}} = \left(\xi_0/g^2\right)\ddot{\xi}\,. \tag{1.66}$$

In dimensionless form, (1.61) reads

$$\ddot{\hat{\xi}} + \varkappa\,\hat{\xi}^{\gamma}\,\dot{\hat{\xi}}^{\beta} + \frac{1+\alpha}{2}\,\hat{\xi}^{\alpha} = 0\,, \qquad \dot{\hat{\xi}}(0) = 1\,, \qquad \hat{\xi}(0) = 0 \tag{1.67}$$

with

$$\varkappa = \varkappa(g) = D_2\left(\frac{1+\alpha}{2\,D_1}\right)^{(1+\gamma)/(1+\alpha)} g^{2(\gamma-\alpha)/(1+\alpha)+\beta}\,. \tag{1.68}$$

None of the terms in (1.67) depends either on material properties or on impact velocity, except for $\varkappa$. Therefore, if the motion of the particles depends on material properties and on impact velocity, it may depend only via $\varkappa$, i.e., in the combination of the parameters as given by (1.68). Hence, any function of the impact velocity, including the coefficient of restitution must be of the form $\varepsilon(g) = \varepsilon[\varkappa(g)]$. A similar result for $\varepsilon \to 0$, $\beta = 1$ and $\alpha = 3/2$ has been obtained in [10].

Hence, if the coefficient of restitution does not depend on the impact velocity g, it is implied that

$$2(\gamma-\alpha) + \beta\;(1+\alpha) = 0\,. \tag{1.69}$$

For small $\dot{\xi}$ a linear dependence of the dissipative force on the velocity seems to be realistic, i.e., $\beta = 1$. Then ε =const. holds true for the following cases:

- For the linear elastic force $F_{\rm el} \propto \xi$, (i.e. $\alpha = 1$) condition (1.69) implies the linear dashpot force $F_{\rm dis} \propto \dot{\xi}$, ($\gamma = 0$).
- For the Hertz law for 3D-spheres (1.24), (i.e. $\alpha = 3/2$), condition (1.69) requires $F_{\rm dis} \propto \dot{\xi}\,\xi^{1/4}$, ($\gamma = \frac{1}{4}$). As far as we can see there is no physical argument to justify this functional form of the dissipative force.

Therefore, we conclude that the assumption ε =const. is in agreement with mechanics of materials only in the case of (quasi-)one-dimensional systems. For three-dimensional spheres it disagrees with basic mechanical laws.

For viscoelastic spheres, according to (1.42), the coefficients are $\alpha = 3/2$, $\beta = 1$, and $\gamma = 1/2$. From (1.68) it follows that

$$\varkappa = \frac{3}{2}\left(\frac{5}{4}\right)^{3/5} A\left(\frac{\rho^{\rm\ eff}}{m}\right)^{2/5} g^{1/5} \tag{1.70}$$

and, therefore,

$$\varepsilon = \varepsilon\left[A\left(\frac{\rho^{\rm\ eff}}{m}\right)^{2/5} g^{1/5}\right]\,. \tag{1.71}$$

If we assume that the function $\varepsilon(g)$ is sufficiently smooth and can be expanded into a Taylor series, and with $\varepsilon(0) = 1$, for small impact velocity the coefficient of restitution reads

$$\varepsilon = 1 - C_1 A\kappa^{2/5} g^{1/5} + C_2 A^2 \kappa^{4/5} g^{2/5} \mp \dots \tag{1.72}$$

where

$$\kappa = \left(\frac{3}{2}\right)^{5/2} \left(\frac{\rho}{m^{\text{eff}}}\right) = \left(\frac{3}{2}\right)^{3/2} \frac{Y\sqrt{R^{\text{eff}}}}{m^{\text{eff}}\left(1-\nu^2\right)} \,. \tag{1.73}$$

The coefficients C_1, $C_2, \dots$ are pure numbers which are given analytically in [24]. Here we give a simple derivation of these coefficients (which is correct for C_1 and C_2 and approximately correct for C_3 and C_4, using the method proposed in [23]).

1.3.3 Coefficient of Restitution for Spheres

1.3.3.1 Small Inelasticity Expansion

Using $d/d\xi = \dot{\hat{\xi}} d/d\hat{\xi}$ the equation of motion for a collision adopts the form

$$\frac{d}{d\hat{\xi}}\left(\frac{1}{2}\dot{\hat{\xi}}^2 + \frac{1}{2}\hat{\xi}^{5/2}\right) = -\varkappa\dot{\hat{\xi}}\sqrt{\hat{\xi}} = \frac{dE(\hat{\xi})}{d\hat{\xi}}\,, \qquad \hat{\xi}(0) = 0\,, \qquad \dot{\hat{\xi}}(0) = 1\,, \tag{1.74}$$

where we introduce the mechanical energy

$$E = \frac{1}{2}\dot{\hat{\xi}}^2 + \frac{1}{2}\hat{\xi}^{5/2}\,. \tag{1.75}$$

The first stage of the collision starts at $\hat{\xi} = 0$ and ends in the turning point of maximal compression $\hat{\xi}_0$. During the second stage, the particles return to $\hat{\xi} = 0$. The energy dissipation during the first stage is given by

$$\int_0^{\hat{\xi}_0} \frac{dE}{d\hat{\xi}} d\hat{\xi} = -\varkappa \int_0^{\hat{\xi}_0} \dot{\hat{\xi}}\sqrt{\hat{\xi}} d\hat{\xi}\,. \tag{1.76}$$

For the evaluation of the right-hand side of (1.76), the dependence $\dot{\hat{\xi}} = \dot{\hat{\xi}}(\hat{\xi})$ is needed. In the case of an elastic collision where the maximal compression is $\hat{\xi}_0 = 1$ (according to the definition of the dimensionless variables) from energy conservation, it follows that

$$\dot{\hat{\xi}}(\hat{\xi}) = \sqrt{1 - \hat{\xi}^{5/2}}\,, \tag{1.77}$$

i.e., $\dot{\hat{\xi}}$ vanishes at the turning point $\hat{\xi} = 1$. For inelastic collisions $\hat{\xi}_0 \lesssim 1$, therefore,

$$\dot{\hat{\xi}}(\hat{\xi}) \approx \sqrt{1 - (\hat{\xi}/\hat{\xi}_0)^{5/2}}\,. \tag{1.78}$$

Using (1.78) the integration in (1.76) may be performed yielding

$$\frac{1}{2}\hat{\xi}_0^{5/2} - \frac{1}{2} = -\varkappa\, b\, \hat{\xi}_0^{3/2} \tag{1.79}$$

where we take into account

$$E(\hat{\xi}_0) = \frac{1}{2}\hat{\xi}_0^{5/2}\,, \qquad\qquad E(0) = \frac{1}{2}\dot{\hat{\xi}}^2(0) = \frac{1}{2} \tag{1.80}$$

and introduce the constant

$$b \equiv \int_0^1 \sqrt{x}\sqrt{1-x^{5/2}}\,dx = \frac{\sqrt{\pi}\,\Gamma\,(3/5)}{5\,\Gamma\,(21/10)}\,. \tag{1.81}$$

Let us define the *inverse collision*, the collision that starts with velocity $\varepsilon\, g$ and ends with velocity g. During the inverse collision, the system gains energy. The maximal compression $\hat{\xi}_0$ is naturally the same for both collisions, since the inverse collision equals the direct collision, except for the fact that time runs in the reverse direction, hence,

$$\frac{dE(\hat{\xi})}{d\hat{\xi}} = +\varkappa\dot{\hat{\xi}}\sqrt{\hat{\xi}}\,, \qquad \dot{\hat{\xi}}(0) = \varepsilon\,, \qquad \hat{\xi}(0) = 0\,. \tag{1.82}$$

This suggests an approximative relation for the inverse collision,

$$\dot{\hat{\xi}}(\hat{\xi}) \approx \varepsilon\,\sqrt{1-(\hat{\xi}/\hat{\xi}_0)^{5/2}}\,, \tag{1.83}$$

with the additional pre-factor ε, which is the initial velocity for the inverse collision.

Integration of the energy *gain* for the first phase of the inverse collision (which equals, up to its sign, the energy loss in the second phase of the direct collision [24]) may be performed just in the same way as for the direct collision, yielding

$$\frac{1}{2}\hat{\xi}_0^{5/2} - \frac{\varepsilon^2}{2} = +\varepsilon\,\varkappa\,b\,\hat{\xi}_0^{3/2}\,, \tag{1.84}$$

where again $E(\hat{\xi}_0) = \hat{\xi}_0^{5/2}/2$ and $E(0) = \varepsilon^2/2$ is used. Multiplying (1.79) by ε and summing it with (1.84), the maximal compression is $\varepsilon = \hat{\xi}_0^{5/2}$. Substituting this into (1.79) we arrive at an equation for the coefficient of restitution

$$\varepsilon + 2\varkappa\,b\,\varepsilon^{3/5} = 1\,. \tag{1.85}$$

The formal solution to this equation may be written as a continuous fraction (which does not diverge in the limit $g \to \infty$):

$$\varepsilon^{-1} = 1 + 2\varkappa\,b(1 + 2\varkappa\,b(1 + \cdots)^{2/5}\cdots)^{2/5} \tag{1.86}$$

For practical applications the series expansion of ε in terms of $\varkappa$ is more appropriate. We return to dimensional units and define the characteristic velocity g^* such that

$$\varkappa \equiv \frac{1}{2b}\left(\frac{g}{g^*}\right)^{1/5}\,, \tag{1.87}$$

with b being defined in (1.81). Using, moreover, the definition (1.68) together with (1.42), which provides the values of D_1 and D_2, the characteristic velocity reads

$$(g^*)^{-1/5} = \frac{\sqrt{\pi}}{2^{1/5}5^{2/5}} \frac{\Gamma(3/5)}{\Gamma(21/10)} \left(\frac{3}{2}A\right) \left(\frac{\rho}{m^{\rm eff}}\right)^{2/5} . \tag{1.88}$$

With this new notation the coefficient of restitution adopts a simple form:

$$\varepsilon = 1 - a_1 \left(\frac{g}{g^*}\right)^{1/5} + a_2 \left(\frac{g}{g^*}\right)^{2/5} - a_3 \left(\frac{g}{g^*}\right)^{3/5} + a_4 \left(\frac{g}{g^*}\right)^{4/5} \mp \cdots , \tag{1.89}$$

with $a_1 = 1$, $a_2 = 3/5$, $a_3 = 6/25 = 0.24$, $a_4 = 7/125 = 0.056$. Rigorous but elaborated calculations [24] show that, while the coefficients a_1 and a_2 are exact, the correct coefficients a_3 and a_4 are: $a_3 \approx 0.315$ and $a_4 \approx 0.161$. The coefficients C_i of the expansion (1.72) can be obtained via

$$C_i = a_i C_1^i = a_i \left(g^*\right)^{-i/5} . \tag{1.90}$$

In particular,

$$C_1 = \frac{\sqrt{\pi}}{2^{1/5}5^{2/5}} \frac{\Gamma(3/5)}{\Gamma(21/10)} , \qquad C_2 = \frac{3}{5} C_1^2 \tag{1.91}$$

and respectively, $C_3 \approx -0.483$, $C_4 \approx 0.285$. The convergence of the series is rather slow, and accurate results can be expected only for small enough g/g^*.

Let us briefly mention a complication of the quasi-static approximation (QSA). During the expansion phase it may happen that the repulsive force according to (1.42) becomes negative, i.e., seemingly the particles attract each other. For the interaction of non-cohesive particles we had, however, excluded attractive forces. This is an artefact, since in reality the particles lose contact already, before completely recovering their spherical shape, i.e., before $\hat{\xi} = 0$ (see [22] for a detailed explanation of this problem). This effect, however, is not in agreement with the QSA. Obviously, (1.42) which is a result of the QSA, derived in [8], is not appropriate to describe the very end of the particle contact. Taking this effect into account we obtain a larger coefficient of restitution as compared with the presented computation [25]. For small dissipation, the correction is rather small. This small correction is neglected here.

1.3.3.2 Padé Approximation

For practical applications, such as molecular dynamics simulations, the expansion (1.89) is of limited value, since it diverges for large impact velocities, $g \to \infty$. It is possible, however, to construct a Padé approximant for ε, based on the above coefficients, which reveals the correct limits, $\varepsilon(0) = 1$ and $\varepsilon(\infty) = 0$. The dependence of $\varepsilon(g)$ is expected to be a smooth monotonically decreasing function, which suggests that the order of the numerator must be smaller than the order of the denominator. The 1-4 Padé-approximant

$$\varepsilon = \frac{1 + d_1 \left(g/g^*\right)^{1/5}}{1 + d_2 \left(g/g^*\right)^{1/5} + d_3 \left(g/g^*\right)^{2/5} + d_4 \left(g/g^*\right)^{3/5} + d_5 \left(g/g^*\right)^{4/5}} \tag{1.92}$$

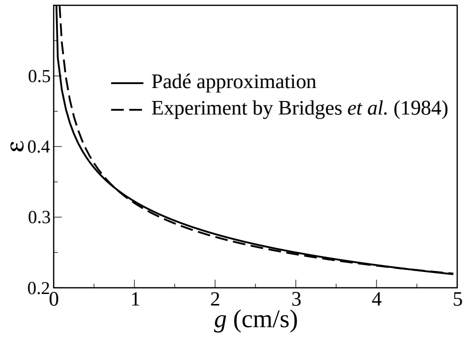

Figure 1.2: Dependence of the coeffi cient of normal restitution on the impact velocity for ice particles. The dashed line is experimental [4], the solid line is the Padé-approximation (1.92) with the constants given by (1.93) and with the characteristic velocity for ice $g^* = 0.32 \text{ cm s}^{-1}$.

satisfies these conditions. Standard analysis (e.g. [3]) yields the coefficients d_k in terms of the coefficients a_k

$$
\begin{aligned}
d_0 &= a_4 - 2a_3 - a_2^2 + 3a_2 - 1 && (1.93)\\
d_1 &= \left[1 - a_2 + a_3 - 2a_4 + (a_2 - 1)(3a_2 - 2a_3)\right]/d_0 && \approx 2.583\\
d_2 &= \left[(a_3 - a_2)(1 - 2a_2) - a_4\right]/d_0 && \approx 3.583\\
d_3 &= \left[a_3 + a_2^2(a_2 - 1) - a_4(a_2 + 1)\right]/d_0 && \approx 2.983\\
d_4 &= \left[a_4(a_3 - 1) + (a_3 - a_2)(a_2^2 - 2a_3)\right]/d_0 && \approx 1.148\\
d_5 &= \left[2(a_3 - a_2)(a_4 - a_2 a_3) - (a_4 - a_2^2)^2 - a_3(a_3 - a_2^2)\right]/d_0 && \approx 0.326
\end{aligned}
$$

Using the characteristic velocity $g^* = 0.32 \text{ cm s}^{-1}$ for ice at very low temperature as a fitting parameter, we compare the theoretical prediction of $\varepsilon(g)$, given by (1.92), with the experimental results [4], see Figure 1.2. The discrepancy with the experimental data at small g follows from the fact that the extrapolation expression, $\varepsilon = 0.32/g^{0.234}$ used by [4] to fit the experimental data has an unphysical divergence at $g \to 0$ and does not imply the failure of the theory for this region. The scattering of the experimental data presented by [4] is large for small impact velocity, according to experimental complications, therefore the fit formula of [4] cannot be expected to be accurate enough for velocities that are too small. For very high velocities the effects, such as brittle failure, fracture and others, may contribute to the dissipation, so that the mechanism of the viscoelastic losses could not be the primary one. In the region of very small velocity, other interactions than viscoelastic ones, e.g., adhesive interactions, may be important.

1.3.4 Coefficient of Restitution for Adhesive Collisions

For very small velocities, when the kinetic energy of the relative motion of colliding particles is comparable with the surface interaction energy at the contact, the adhesive forces play an important role in collision dynamics – they may change the coefficient of restitution qualitatively. Indeed, as described above, adhesive particles in contact are compressed even for vanishing external load, i.e., a tensile force must be applied to separate the particles. Therefore, at the second stage of the collision, the separating particles must overcome a barrier due to the attractive interaction, which keeps them together. The work against this tensile force reduces the kinetic energy of the relative motion after the collision, that is, it reduces the effective coefficient of restitution. For small impact velocity the kinetic energy of the relative motion may be too small to overcome the attractive barrier, i.e., the particles stick together after the collision, corresponding to $\varepsilon = 0$.

From these arguments it follows that the description of particle collisions by pure viscoelastic interaction has a limited range of validity, not only for large impact rate when plastic deformation becomes important, but also for small impact rata due to adhesion. A simplified analysis of adhesive collisions is presented in [5] to estimate the influence of adhesive forces on the coefficient of restitution. It allows to estimate the range of validity of the viscoelastic collision model.

We assume that the JKR theory is adequate for the given system parameters. We also assume that the adhesion is small and that the adhesive interactions may be neglected when the force between the particles is purely repulsive. Hence, we take into account the influence of adhesive interaction only when the total force is attractive, that is when the force is mainly determined by adhesion. This happens at the very end of the collision. We also neglect the additional dissipative forces, which arise due to the adhesive interaction and assume that all dissipation during the collision may be attributed to the viscoelastic interactions.

At the second stage of a collision, when the particles move away from each other they pass the point where the contact area is a_0 and the total force vanishes. As the particles move away further, the force becomes negative, until it reaches, at $a = a_{\rm sep}$, the maximum negative value $F = F_{\rm sep}$, Eq. (1.51). At this point the contact of the particles is terminated and they separate. According to our assumption, the work of the tensile force which acts against the particles, separation reads

$$W_0 = \int_{\xi(a_0)}^{\xi(a_{\rm sep})} F(\xi) d\xi = \int_{a_0}^{a_{\rm sep}} F(a) \frac{d\xi}{da} da\,. \tag{1.94}$$

Using (1.49) for the total force $F(a)$, (1.48) for the compression, which allows one to obtain $d\xi/da$, and (1.50), (1.52) for a_0 and $a_{\rm sep}$, we obtain the work of the tensile forces,

$$W_0 = q_0 \left(\pi^5 \gamma^5 D^2 R^4\right)^{1/3}\,, \tag{1.95}$$

with the constant

$$q_0 = \frac{1}{10} \left(2^{1/3}\, 3 - 1\right) 3^{2/3}\,. \tag{1.96}$$

Close to the end of the collision, just before the tensile forces start to act, the relative velocity is $g' = \varepsilon g$. The final velocity g'', when the particles completely separate from each other, may be found from the conservation of energy:

$$\frac{1}{2} m^{\rm eff} \left(g'\right)^2 - \frac{1}{2} m^{\rm eff} \left(g''\right)^2 = W_0\,. \tag{1.97}$$

From the latter equation we obtain the coefficient of restitution for the adhesive collision, $\varepsilon_{\rm ad}$,

$$\varepsilon_{\rm ad}(g) = \frac{g''}{g} = \frac{\sqrt{\varepsilon^2(g)g^2 - 2W_0/m^{\rm eff}}}{g}\,, \tag{1.98}$$

where $\varepsilon(g)$ is the coefficient of restitution without the adhesive interaction. Hence we obtain the condition for the validity of the viscoelastic collision model,

$$\varepsilon(g)g \gg \sqrt{\frac{2W_0}{m^{\rm eff}}}\,. \tag{1.99}$$

The threshold impact velocity $g_{\rm st}$ which separates the restitutive ($g > g_{\rm st}$) from the sticking ($g < g_{\rm st}$) collisions, may be obtained from the solution of the equation

$$\frac{1}{2}m^{\rm eff}\varepsilon^2(g)g^2 = W_0\,. \tag{1.100}$$

Using (1.72) we obtain for viscoelastic spheres, in the leading-order approximation, with respect to the small dissipative parameter A:

$$g_{\rm st} = \sqrt{\frac{2W_0}{m^{\rm eff}}}\left[1 + C_1 A\kappa^{2/5}\left(\frac{2W_0}{m^{\rm eff}}\right)^{1/10}\right]\,. \tag{1.101}$$

For head-on collisions (vanishing tangential component of the impact velocity) the colliding particles stick together if $g < g_{\rm st}$. In this case, after the collision, the particles form a joint particle of mass $m_1 + m_2$.

1.4 Conclusion

We have considered the collision of particles in granular matter with respect to viscoelastic and adhesive interaction. Thus, the elastic contribution due to the classical Hertz theory is complemented by the simplest model for dissipative material deformation, where the viscous stress is linearly related to the strain rate. Moreover, quasi-static approximation was assumed, i.e., the impact velocity is much smaller than the speed of sound in the material and the viscosity relaxation time is much smaller than the duration of the collision. Using these approximations, we obtained the general solution for the contact problem for convex viscoelastic bodies. The validity of this model is violated for large impact velocity due to plastic deformations and also for very small impact velocity due to surface forces. We have discussed the available models of adhesive interaction. For the model by Johnson, Kendall and Roberts [17] which has been shown to be accurate in a range of parameters of practical interest, the additional dissipation arising due to adhesive forces have been estimated. From the comparison of the force contribution due to pure viscoelastic interaction and the contribution due to adhesion, we have estimated the range of validity of the viscoelastic model. For head-on collisions we have also estimated the marginal value of the impact velocity, which discriminates restitutive and sticking collisions.

References

[1] M. Abramowitz and A. Stegun. *Handbook of Mathematical Functions.* Dover Publications, New York, 1965.

[2] P. Attard and J. L. Parker. Deformation and adhesion of elastic bodies in contact. *Phys. Rev. A*, 46:7959, 1992.

[3] G. A. Baker. *The Padé Approximant in Theoretical Physics*. Academic Press, New York, 1970.

[4] F. G. Bridges, A. Hatzes, and D. N. C. Lin. Structure, stability and evolution of Saturn's rings. *Nature*, 309:333, 1984.

[5] N. V. Brilliantov. Viscoelastic interactions of adhesive particles. *Preprint*, 2004.

[6] N. V. Brilliantov and T. Pöschel. *Kinetic Theory of Granular Gases*. Oxford University Press, Oxford, 2004.

[7] N. V. Brilliantov, F. Spahn, J.-M. Hertzsch, and T. Pöschel. The collisions of particles in granular systems. *Physica A*, 231:417, 1996.

[8] N. V. Brilliantov, F. Spahn, J.-M. Hertzsch, and T. Pöschel. Model for collisions in granular gases. *Phys. Rev. E*, 53:5382, 1996.

[9] B. V. Derjaguin, V. M. Muller, and Yu. Toporov. Effect of contact deformations on the adhesion of particles. *J. Colloid Interface Sci.*, 53:314, 1975.

[10] E. Falcon, C. Laroche, S. Fauve, and C. Coste. Behavior of one inelastic ball bouncing repeatedly off the ground. *Eur. Phys. J. B*, 3:45, 1998.

[11] J. A. Greenwood. Adhesion of elastic spheres. *Proc. R. Soc. Lond. A*, 453:1277, 1977.

[12] J. A. Greenwood and K. L. Johnson. An alternative to the maugid model of adhesion between elastic spheres. *J. Phys. D: Appl. Phys.*, 31:3279, 1998.

[13] G. Haiat, M. C. Phan-Huy, and E. Barthel. The adhesive contact of viscoelastic spheres. *J. Mechanics and Physics of Solids*, 51:69, 2003.

[14] H. Hertz. Über die Berührung fester elastischer Körper. *J. f. reine u. angewandte Math.*, 92:156, 1882.

[15] J.-M. Hertzsch, F. Spahn, and N. V. Brilliantov. On low-velocity collision of viscoelastic particles. *J.Phys. II France*, 5:1725, 1995.

[16] B. D. Hughes and L. R. White. The implications of elastic deformation on the direct measurement of surface forces. *J. Chem. Soc., Faraday Transactins I*, 76:963, 1980.

[17] K. L. Johnson, K. Kendall, and A. D. Roberts. Surface energy and contact of elastic solids. *Proc. R. Soc. London Ser. A*, 324:301, 1971.

[18] L. D. Landau and E. M. Lifshitz. *Theory of Elasticity*. Oxford University Press, Oxford, 1965.

[19] D. Maugis. Adhesion of spheres: the JKR-DMT transition using a Dugdale model. *J. Colloid Interface Sci.*, 150:243, 1992.

[20] V. M. Muller, V. S. Yuschenko, and B. V. Derjaguin. On the influence of molecualar forces on the deformation of an elastic sphere and its sticking to a rigid plane. *J. Colloid Interface Sci.*, 77:91, 1980.

[21] V. M. Muller, V. S. Yuschenko, and B. V. Derjaguin. General theoretical consideration of the influence of surface forces on contact deformation. *J. Colloid Interface Sci.*, 92:92, 1983.

[22] T. Pöschel and T. Schwager. *Computational Granular Dynamics: Models and Algorithms*. Springer, Berlin, 2004.

[23] R. Ramírez, T. Pöschel, N. V. Brilliantov, and T. Schwager. Coefficient of restitution for colliding viscoelastic spheres. *Phys. Rev. E*, 60:4465, 1999.

[24] T. Schwager and T. Pöschel. Coefficient of restitution of viscous particles and cooling rate of granular gases. *Phys. Rev. E*, 57:650, 1998.

[25] T. Schwager and T. Pöschel. Duration of contact and coefficient of restitution. *preprint*, 2004.

[26] U. D. Schwarz. A generalized analytical model for the elastic deformation of an adhesive contact between a sphere and a flat surface. *J. Colloid Interface Sci.*, 261:99, 2003.

[27] J. R. Smith, G. Bozzolo, A. Banerjea, and J. Ferrante. Avalanche in adhesion. *Phys. Rev. Lett.*, 63:1269, 1989.

[28] F. Spahn, N. Albers, M. Sremcevic, and C. Thornton. Kinetic description of coagulation and fragmentation in dilute granular particle ensembles. *Europhys. Lett.*, 67:545, 2004.

[29] D. Tabor. Surface forces and surface interactions. *J. Colloid Interface Sci.*, 58:2, 1977.

[30] Y. Taguchi. Powder turbulence: Direct onset of turbulent flow. *J. Physique*, 2:2103, 1992.

[31] T. Tanaka, T. Ishida, and Y. Tsuji. Direct numerical simulation of granular plug flow in a horizontal pipe: The case of cohesionless particles (in japanese). For an english presentation of this work see [30]. *Trans. Jap. Soc. Mech. Eng.*, 57:456, 1991.

[32] S. P. Timoshenko. *Theory of Elasticity*. McGraw-Hill, New York, 1970.